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STOCHASTIC INTEGRALS OF WEIGHTED EMPIRICAL PROCESSES AND AN APPLICATION TO THE LIMITING DISTRIBUTION OF LINEAR RANK STATISTICS UNDER ALTERNATIVES

by

Navaratna S. Rajaram

Submitted to the faculty of the Graduate School in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics

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Let $X_{N1}, \dots, X_{NN}$ be independent random variables with continuous cumulative distribution functions $F_{N1}, \dots, F_{NN}$ respectively. Consider the rank order statistics

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20 Abstract

$$S_N = \sum_{i=1}^{N} C_{Ni} a_N (R_{Ni}),$$

where $R_{N1},\ldots,R_{NN}$ are the respective ranks of $X_{N1},\ldots,X_{NN}$; $C_{N1},\ldots,C_{NN}$ are known constants, and $a_{N(1)},\ldots,a_{N(N)}$ are scores generated by a known function $\varphi(t)$, $0 < t < 1$ in a specified manner. Asymptotic normality of S_N is obtained by expressing it as the stochastic integral of suitably defined empirical and weighted empirical processes, and using techniques of convergence of stochastic processes and reproducing Kernel Hilbert spaces.

An alternative approach to the same problem is provided by using stability results from classical probability theory. A multivariate extension is also given.

The phenomenon of Gaussian noise is examined by viewing it as the limiting action of a sequence of disturbances caused by finitely many random variables acting on a certain function space. A central limit theorem for white noise is proved. A connection between such noise phenomenon and the asymptotic behavior of certain rank statistics is exposed.

TABLE OF CONTENTS :

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INTRODUCTION :

This paper is motivated by the desire to extend methods of Stochastic Processes to problems of the regression type encountered in Nonparametric Statistics, under a large class of alternatives. A brief discription of the evolution of the problem is given below.

In 1958, Chernoff and Savage published their basic paper on the asymptotic normality of certain nonparametric statistics which stimulated a great deal of research activity in the area. In 1968, Pyke and Shorack obtained a proof of the Chernoff-Savage theorem by studying the limiting behavior of an empirical process. Subsequent to their work, although much work has been done in the area of stochastic processes which arise in Statistics, problems of the regression type have for the most part remained outside the purview of such methods. The method of attack for such problems has so far been along the lines of Hàjek (1961, 1968), Hoeffding (1973), Puri and Sen (1969), Dupač and Hàjek (1969), to name a few. The most general results in this direction are to be found in Hàjek (1968).

In this paper, theorems of the type proved in Hàjek (1968) are proved by methods of Stochastic Processes. During the course of the study it was found that existing techniques of weak convergence of stochastic porcesses, such as for instance found in Pyke and Shorack (1968) or in Billingsley (1968) lead to certain insuperable technical problems. The classical

weak convergence techniques for such problems can briefly be described as follows:

Suppose it is required to find the limiting distribution of a functional $f(X_1,\ldots,X_N)$ of observations $X_1,\ldots,X_N$. First, a suitable empirical process $X_N(t)$ based on the observations $X_1,\ldots,X_N$ is defined. Then, conditions for convergence of $X_N(t)$ to another (usually Gaussian) process $X(t)$ are obtained. Then with some continuity restrictions on the functional f, the limiting distribution of $f(X_N(t))$ is the same as the distribution of the random variable $f(X(t))$. Convergence of $X_N(t)$ to $X(t)$ involves two problems. First of which is the usually simpler problem of establishing the convergence of the finite dimensional distributions of $X_N(t)$ to those of $X(t)$. Second, to obtain certain restrictions on the sample path fluctuations of $X_N(t)$. If the number of different distributions involved becomes large, as it does in the problem considered in this paper, any analysis of the sample paths becomes extremely involved.

In view of this last difficulty, it was found necessary to develop a technique based on the second order properties of empirical processes, thereby circumventing the need for sample path analysis entailed by the classical weak convergence methods. This however leads to technical difficulties of a different kind in that certain non-random estimates have to be obtained for the remainder terms. During the course of estimating these remainder terms, it was found that a fairly

simple proof of the asymptotic normality of Hàjek type statistic could be given along the lines of the original proof of the Chernoff-Savage theorem. This proof is given in Chapter II.

In Chapter I, the second order stochastic integral is defined and conditions for the existence of such an integral are proved. Next the empirical and the weighted empirical process are defined and their reproducing kernels found. Then criteria for second order convergence of these processes to Gaussian processes are obtained. Then variance inequalities are obtained for stochastic integrals of these processes. These results are in turn used to obtain the asymptotic distribution of a simple linear rank statistic.

In Chapter II, a different proof is given for the asymptotic distribution of such a statistic. Also the remainder terms are estimated.

In Chapter III, a multivariante extension is given.

It is an interesting fact that stochastic integrals of the type considered here also appear in an entirely different context in E. Parzen's pioneering work on Time Series Analysis. But the kernels found here and the Hilbert spaces which they span, apart from being useful are also of independent interest. In Chapter I, extensive use is made of the very elegant exposition of second order properties of stochastic processes given in Loève (1963). Some very interesting results concerning convergence in distribution of stochastic integrals can be found in a paper by M. Brown (1970).

In Chapter IV, a classification and properties of noise phenomenon caused by finitely many random variables as well as their convergence to Gaussian noise are given. Further a relationship between noise phenomenon and the distributions of rank statistics is exposed.

CHAPTER I

Second Order Approach to the Limiting Distribution of Simple Linear Rank Statistics:

1.1 Preliminaries: Let $X_{N1},\ldots,X_{NN}$ be a sequence of independent variables with the corresponding continuous distribution functions $F_{N1},\ldots,F_{NN}$. Let R_{Ni} be the rank of X_{Ni} among $X_{N1},\ldots,X_{NN}$.

i.e. $R_{Ni} = (\text{Number of } X_{Nj} \leq X_{Ni})$; $j = 1,\ldots,N$.

Consider a simple linear rank statistic defined by

$$S_N = \sum_{i=1}^{N} C_{Ni}\, \varphi\left(\frac{R_{Ni}}{N+1}\right) \tag{1.1.1}$$

where

$C_{N1},\ldots,C_{NN}$ are known constants and

$\varphi : (0,1) \to \mathbb{R}$ is a known function.

Statistics of this type play a very important role in nonparametric theory; the Wilcoxin and the normal scores statistics being the best known. We are interested in the limiting normality of S_N. Hajek [12] has obtained the limiting normality of S_N under mild conditions on φ. But his proofs are rather involved and we will develop a

technique using stochastic integrals of suitable second order processes. This approach is motivated by the methods of Pyke and Shorack [23] which we will briefly describe below.

In the special case when $C_{N1} = C_{N2} = \ldots = C_{Nm} = \frac{1}{m}$,

$$C_{Nm+1} = \ldots = C_{NN} = 0 ,$$

$$F_{N1} = F_{N2} = \ldots = F_{Nm} = F \text{ and}$$

$$F_{Nm+1} = \ldots = F_{NN} = G$$

S_N reduces to the Chernoff-Savage two-sample case.

In this case, Pyke and Shorack consider the weak convergence of the process

$$\mathcal{L}_N(t) = \sqrt{N}\,(F_m(H_N^{-1}(t)) - F(H^{-1}(t))) \quad \text{where} \tag{1.1.2}$$

F_m is the empirical distribution function of $X_{N1},\ldots,X_{Nm}$, H_N is the empirical distribution function of the entire sample and $H = \frac{m}{N} F + \frac{N-m}{N} G$.

For the more general S_N defined in (1.1.1) , it is thus natural to study the weighted empirical process

$$\mathcal{L}_{N,C}(t) = \sum_{i=1}^{N} C_{Ni}\{I(X_{Ni} \le H_N^{-1}(t)) - F_{Ni}(H^{-1}(t))\} \tag{1.1.3}$$

where $I(x \le y) = 1$ when $x \le y$, $= 0$ when $x > y$.

Unfortunately this process is much too complex and classical weak convergence methods entail a detailed study of the sample path fluctuations of the process. We will develop a technique which uses only the second order properties

and obtain convergence of S_N without an analysis of sample paths.

The possibility of such an approach was suggested by Brown [5] and Parzen [18] . Such techniques are quite well known in Times Series Analysis but have not been used in limiting distribution theory. In the next section, we will give the necessary definitions and results.

1.2. Stochastic Integral of a Second Order Stochastic Process :

Let $\{X(t) : t \in T\}$ be a real-valued second order stochastic process with kernel $K(s,t) = E[X(s)X(t)]$. We will always assume that T is an interval of real numbers.

Let $\{Y(t) : t \in T\}$ be another real-valued second order stochastic process with the kernel $R(s,t) = E(Y(s)Y(t))$.

Let $T = [a,b]$ be an interval. We define the second order stochastic integral of $X(t)$ w.r. to $Y(t)$ as follows:

Let $D_T : a = t_1 < \ldots < t_{n+1} = b$.

Define the random step function

$$(1.2.1) \qquad \int_T X_{D_T}(t)\,dY(t) = \sum_{k=1}^{N} X_k\{Y(t_{k+1}) - Y(t_k)\}$$

where $X_k = X(t_k')$, $t_k \le t_k' \le t_{k+1}$.

Then the stochastic integral of $X(t)$ w.r. to $Y(t)$ is

$$(1.2.2) \qquad \int_T X(t)\,dY(t) = \lim_{\max\limits_{1\le k\le n}|t_{k+1}-t_k|\to 0} \text{q.m.} \int_T X_{DT}(t)\,dX(t)$$

if it exists; where q.m. stands for the quadratic mean.

In certain cases it might be necessary to define it as an improper integral on $(a,b) = T$.

Then we take $T' = [\alpha, \beta]$ and first define

$$(1.2.3) \qquad \int_{T'} X(t)\, dY(t), \quad a < \alpha < \beta < b$$

and then define

$$(1.2.4) \qquad \int_T X(t) d\, Y(t) = \lim_{\substack{\alpha \to a \\ \beta \to b}} \text{q.m.} \int_{T'} X(t) d\, Y(t)$$

if it exists.

<u>Remark 1</u> : These integrals depend only on the "increments" of $Y(t)$ and not on $Y(t)$ itself.

<u>Remark 2</u> : In most applications $Y(t)$ will be a non-random function $g(t)$ and the kernel $R(s,t) = g(s)g(t)$. In such a case the stochastic integral (1.2.2) reduces to

$$(1.2.5) \qquad \int_T X(t)\, d\, g(t)$$

Since $X(t)$ is a second order process, it is well known that it spans a Hilbert space with $K(s,t)$ as its reproducing kernel. Thus stochastic integrals of the type (1.2.5) represent a linear functional on the Reproducing Kernel Hilbert Space spanned by $X(t)$ or $K(\ ,\ )$. See Loeve [17] or Parzen [18] , for some properties of such a space. Also see appendix to this chapter.

Let $\mathcal{L}_2(X)$ be the Hilbert space spanned by $X(t)$. We can identify $\mathcal{L}_2(X)$ with a Hilbert space $\mathcal{H}_K$ as follows:

Define a mapping

$$\text{(1.2.7)} \qquad \Lambda : \mathcal{L}_2(X) \to \mathcal{H}_K$$
$$X(t) \to K_t \quad \text{where}$$
$$K_t(s) = K(s,t) \ , \quad s \in T$$

We define the inner product on $\mathcal{H}_K$ as

$$\text{(1.2.8)} \qquad (K_t \ , K_s) = (X(t) \ , X(s)) = E(X(t)\,X(s)) = K(s,t)$$

$$\text{(1.2.9)} \qquad \text{i.e.} \quad (X(s), X(t)) = (\Lambda X(s) \ , \Lambda X(t)) \ .$$

Since $X(t)$ is dense in $\mathcal{L}_2(X)$ (since $X(t)$ spans $\mathcal{L}_2(X)$) , we can regard $\mathcal{H}_K$ as being spanned by the family $\{K_t : t \in T\}$. Λ , by (1.2.8) preserves inner products, $\|\Lambda\| = 1$ and is therefore continuous.

In (1.2.6) if we let

$$y = \int_T X(t) \, dg(t) \ , \quad y \in \mathcal{L}_2(X) \text{ by definition}$$

of the stochastic integral and

$$(1.2.10)\qquad \Lambda y = \Lambda \int_T X(t)\, d\, g(t) = \int_T (\Lambda X(t))\, d\, g(t)$$

$$= \int_T K_t dg(t) \quad \text{whereby,}$$

$$\Lambda y : T \to \mathbb{R} \quad \text{such that}$$

$$(1.2.11)\qquad (\Lambda y)(s) = (\int_T^0 K_t dg(t))(s) = \int_T K(s,t)dg(t) \quad .$$

We then have the following interesting result.

<u>Proposition 1.2.1</u> : If $f \in \mathcal{H}_K$, then $f(s) = (f,K_s)$.

<u>Proof</u>: Let $f(s) = \int_T K(s,t)dg(t)$.

Then $f = \Lambda \int_T X(t)dg(t)$

$(f , K_s) = (\Lambda \int_T X(t)dg(t), \Lambda X(s))$

$= (\int_T X(t)dg(t) , X(s))$ since Λ is inner product preserving.

Thus

$$(f,K_s) = (\int_T X(t)dg(t) , X(s))$$

$$= E[X(s) \int_T X(t)dg(t)] = E \int_T X(s)X(t)dg(t)$$

$$= \int_T E(X(s)X(t))dg(t) = \int_T K(s,t)dg(t)$$

$= f(s)$ which completes the proof.

$K(\ ,\ )$ is called the Reproducing Kernel of the Hilbert space $\mathcal{H}_K$.

See Aronszajn [1] , for a comprehensive discussion of Reproducing Kernels. Yošida [11] has a different approach but a less detailed discussion.

In general, the class of processes $Y(t)$ (on functions $g(t)$) for which stochastic integrals exist depends on the process $X(t)$; to be more precise, it depends on the corresponding kernels as the following theorem shows.

__Theorem 1.2.2__ : __Let the second order stochastic process $X(t)$ with the kernel $K(s,t)$ be independent of the "increments" $\Delta Y(t)$ of the second order stochastic process $Y(t)$ with the kernel__ $R(s,t)$. __Then the stochastic integral__

$\int_T X(t)dY(t)$ __exists in the sense of definition__

(1.2.2) __if and only if__

$\int_T\int_T K(s,t)d_{12}R(s,t)$ __exists as a Riemann-Stieltjes integral.__

__Proof__ : See Loeve [17] , page 472.

In applications, $Y(t)$ will be a non-random function and the preceding theorem degenerates to the following special case.

<u>Corollary</u> : <u>The stochastic integral</u> $\int_T X(t)dg(t)$ <u>exists if and only if</u> $\int_T\int_T K(s,t)dg(s)dg(t)$ <u>exists as a Riemann-Stieltjes (perhaps improper) integral</u>.

<u>Proof</u> : Immediate noting that $E(g(s)g(t)) = g(s)g(t)$ and substituting in the preceding theorem.

<u>Remark</u> : In general the second order properties do not completely specify a stochastic process. There is however a very important special case : namely when $X(t)$ is a normal (Gaussian) process. Normal processes also satisfy the following useful property :

<u>Theorem 1.2.3</u> : <u>Normality is preserved under integration in q.m</u>. <u>Equivalently, stochastic integrals of Gaussian processes are Gaussian random variables</u>.

<u>Proof</u> : See Loeve [17] .

1.3. <u>Second Order Properties of Weighted Empirical Processes</u>:

Let $X_{N1}, X_{N2}, \ldots, X_{NN}$ be independent random variables with corresponding distribution functions $F_{N1}, F_{N2}, \ldots, F_{NN}$ which are continuous.

We define

$$F_N = \frac{1}{N} \sum_{i=1}^{N} F_{Ni} \tag{1.3.1}$$

Let $C_{N1}, C_{N2}, \ldots, C_{NN}$ be known constants.

We shall investigate the second order properties of the two processes.

$$(1.3.2)\quad X_N(t) = \sqrt{N}\,\{\frac{1}{N}\sum_{i=1}^{N}[I(X_{Ni} \le F_N^{-1}(t) - F_{Ni}(F_N^{-1}(t))]\}$$

$$= \sqrt{N}\,\{\frac{1}{N}\sum_{i=1}^{N} I(X_{Ni} \le F_N^{-1}(t)) - t\}$$

$$(1.3.3)\quad Y_N(t) = \frac{1}{s_N}\{\sum_{i=1}^{N} C_{Ni}[I(X_{Ni} \le F_N^{-1}(t))-F_{Ni}(F_N^{-1}(t))]\} .$$

Where $s_n > 0$ is a chosen normalizing constant and the domain of each of the processes is $T = (0,1)$; i.e. $0 < t < 1$.

$I(\ )$ denotes the indicator function.

The process $X_N(t)$ is of course the well known empirical process and has been extensively studied. We will only state the needed results for $X_N(t)$ and quote the appropriate references. It is however in the behaviour of $Y_N(t)$ that we are chiefly interested. The process $Y_N(t)$ as well as the process $\mathcal{L}_{N,C}(t)$ defined in (1.1.3) are discussed by Koul and Koul and Staudte (see, [15] and [16]) but our approach is significantly different.

<u>Remark</u> : We have used $F_N = \frac{1}{N}\sum_{i=1}^{N} F_{Ni}$, in place of H in most of the literature to emphasize its dependence on N , which is needed here.

<u>Lemma 1.3.1</u> : $E\,X_N(t) = E\,Y_N(t) = 0$.

(1.3.4) $E(X_N(s)\,X_N(t)) = K_N(s,t) = s\wedge t - \frac{1}{N}\sum_{i=1}^{N} F_{Ni}(F_N^{-1}(s))F_{Ni}(F_N^{-1}(t))$

(1.3.5) $E(Y_N(s)\,Y_N(t)) = R_N(s,t) =$

$$= \frac{1}{s_N^2}\sum_{i=1}^{N} c_{Ni}^2\{F_{Ni}(F_N^{-1}(s))\wedge F_{Ni}(F_N^{-1}(t)) - F_{Ni}(F_N^{-1}(s))F_{Ni}(F_N^{-1}(t))\}$$

<u>Proof</u> : Proof of $E\,X_N(t) = 0 = E\,Y_N(t)$ is obvious. Proof of (1.3.4) is well-known (see for example: Shorack [24]). We will prove (1.3.5).

Without loss of generality, let $s < t$. Let $F_N^{-1}(s) = u$, $F_N^{-1}(t) = v$. Hence $u < v$.

Then, $Y_N(s)\,Y_N(t)$

$$= \frac{1}{s_N^2}\sum_{i,j=1}^{N} c_{Ni}c_{Nj}\{I(X_{Ni}\le u) - F_{Ni}(u)\}\{I(X_{Nj}\le v) - F_{Nj}(v)\}$$

$$= \frac{1}{s_N^2}\sum_{i=1}^{N} c_{Ni}^2\{I(X_{Ni}\le u) - F_{Ni}(u)\}\{I(X_{Ni}\le v) - F_{Ni}(v)\}$$

$$+ \frac{1}{s_N^2}\sum_{i\ne j} c_{Ni}c_{Nj}\{I(X_{Ni}\le u) - F_{Ni}(u)\}\{I(X_{Nj}\le v) - F_{Nj}(v)\} .$$

Observe that :

(i) When $i \neq j$ $I(X_{Ni}\le u)$ is independent of $I(X_{Nj}\le v)$ and $E[I(X_{Ni}\le u) - F_{Ni}(u)] = 0$. Hence the expectation of the second summation vanishes.

(ii) $E\, I(X_{Ni} \le u)\, I(X_{Ni} \le v) = F_{Ni}(u) = F_{Ni}(u) \wedge F_{Ni}(v)$.

For the cross product terms

$$E[I(X_{Ni} \le u) F_{Ni}(v) + I(X_{Ni} \le (v)) F_{Ni}(u)]$$

$$= 2F_{Ni}(u)\ F_{Ni}(v)\ .$$

Combining (i) and (ii) and noting that $u = F_N^{-1}(s)$, $v = F_N^{-1}(t)$, we have

$$R_N(s,t) = \frac{1}{s_N^2} \sum_{i=1}^{N} C_{Ni}^2 \{F_{Ni}(F_N^{-1}(s)) \wedge F_{Ni}(F_N^{-1}(t)) - F_{Ni}(F_N^{-1}(s)) F_N^{-1}(F_N^{-1}(t)\}$$

which completes the proof.

Criterion for $\mathfrak{L}_2$-convergence:

Definition : We will say that a second order stochastic process $X_N(t)$ converges in $\mathfrak{L}_2$ to another (second order) process $X(t)$ if and only if

$$\|X_N(t) - X(t)\| = \{\text{Var } (X_N(t) - X(t)\}^{\frac{1}{2}} \to 0\ ,\ \forall t \quad \text{as } N \to +\infty$$

We then write

$$X_N(t) \xrightarrow{\mathfrak{L}_2} X(t) \quad \text{or}$$

$$\|X_N(t) - X(t)\| \to 0\ .$$

We will now obtain conditions for $\mathfrak{L}_2$-convergence of $Y_N(t)$ defined in (1.3.3).

Theorem 1.3.2 : The second order process $Y_N(t)$ defined in (1.3.3) with the kernel $R_N(s,t)$ defined in (1.3.5) converges in $\mathcal{L}_2$ to a second order process $Y(t)$ if and only if, for each $t \in T$,

$$(1.3.6)\quad \lim_{\substack{M\to+\infty \\ N\to+\infty}} \frac{1}{s_N s_M} E\Big[\sum_{\substack{1\le i\le N \\ 1\le j\le M}} C_{Ni}C_{Mj}\{I(X_{Ni} \le F_N^{-1}(t)) - F_{Ni}(F_N^{-1}(t))\} \times\{I(X_{Mj} \le F_M^{-1}(t)) - F_{Mj}(F_M^{-1}(t))\}\Big]$$

$$= f(t) \text{ exists.}$$

Further, if (1.3.6) holds, the kernel of $X(t)$ is

$$(1.3.7)\qquad E\,Y(s)Y(t) = R(s,t) = \lim_{N\to+\infty} R_N(s,t) .$$

In most statistical applications, we have a continuous sampling situation; that is to say if $N \ge M$, we have $X_{N1} = X_{M1},\ldots,X_{NM} = X_{MM}$. (This is the Hajek set up). Then the above theorem simplifies to the following corollary:

Corollary 1.3.3 : Let $N \ge M$ and the sequences be such that, $X_{Mj} = X_{Nj}$, $j \le M$. The process $Y_N(t)$ with the kernel $R_N(s,t)$ be as defined earlier. Then $Y_N(t)$ converges to a process $Y(t)$ with the kernel $R(s,t) = \lim_{N\to+\infty} R_N(s,t)$ if and only if,

$$(1.3.8)\quad \lim_{\substack{M\to+\infty\\N\to+\infty}} \frac{1}{s_N s_M} \sum_{i=1}^{M} C_{Ni}C_{Mi}[F_{Mi}(F_M^{-1}(t)) \wedge F_{Ni}(F_N^{-1}(t)) - F_{Mi}(F_M^{-1}(t))F_{Ni}(F_N^{-1}(t))]$$

exists and is finite for each t .

<u>Proof of Theorem 1.3.2</u> : By Theorem A, page 469 of Loève, process $Y_N(t) \xrightarrow{\mathcal{L}_2} Y(t)$ if and only if $\lim_{\substack{M\to+\infty\\N\to+\infty}} EY_M(t)Y_N(t)$ exists and is finite for each t . Then $\lim_{N\to+\infty} R_N(s,t) = R(s,t)$ is the kernel of $Y(t)$.

Since we have made no assumptions about the sequences X_{Ni} and X_{Nj} , no further simplification is possible.

<u>Proof of Corollary 1.3.3</u> : Note that $M \leq N$ and hence $F_{M1} = F_{N1}, \ldots, F_{MM} = F_{NM}$. But $F_N = \frac{1}{N}\sum_{i=1}^{N} F_{Ni}$ and $F_M = \frac{1}{M}\sum_{j=1}^{M} F_{nj}$ could be different. Also C_{Ni} may not equal C_{Ni} and s_N and s_M can be different.

In order to simplify the notation we shall write $F_N^{-1}(t) = u$, $F_M^{-1}(t) = v$.

Then,

$$(1.3.9)\quad Y_N(t)\, Y_M(t) =$$

$$= \frac{1}{s_N s_M} \sum_{\substack{1\leq i\leq N\\1\leq j\leq M}} C_{Ni}C_{Mj}\{I(X_{Ni} \leq u) - F_{Ni}(u)\}\{I(X_{Mj} \leq v) - F_{Mj}(v)\}$$

$$= \frac{1}{s_N s_M} \sum_{i=j=1}^{M} C_{Ni}C_{Mi}\{I(X_{Ni} \leq u) - F_{Ni}(u)\}\{I(X_{Ni} \leq v) - F_{Ni}(v)\}$$

$$+ \frac{1}{s_N s_M} \sum_{i \neq j} c_{Ni} c_{Mj} \{I(X_{Ni} \le u) - F_{Ni}(u)\}\{I(X_{Mj} \le v) - F_{Mj}(v)\} .$$

In the first term, we have used the fact that $X_{Ni} = X_{Mi}$ and $F_{Ni} = F_{Mi}$ for $i \le M$. We observe the following:

(i) when $i \neq j$, X_{Ni} and X_{Nj} are independent. Hence $E[\{I(X_{Ni} \le u) - F_{Ni}(u)\}\{I(X_{Nj} \le v) - F_{Mj}(v)\}]$

$$= E[I(X_{Ni} \le u) - F_{Ni}(u)]\ E[I(X_{Nj} \le v) - F_{Nj}(v)] = 0 .$$

(ii) when $i = j$,

$$(1.3.10) \quad [I(X_{Ni} \le u) - F_{Ni}(u)][I(X_{Ni} \le v) - F_{Ni}(v)]$$

$$= I(X_{Ni} \le u) I(X_{Ni} \le v) - I(X_{Ni} \le u) F_{Ni}(v)$$

$$- I(X_{Ni} \le v) F_{Ni}(u) + F_{Ni}(u) F_{Ni}(v) .$$

In (3.10), we observe that

$$E\{I(X_{Ni} \le u) I(X_{Ni} \le v)\} = F_{Ni}(u \wedge v) = F_{Ni}(u) \wedge F_{Ni}(v) .$$

Taking expectations,

$$(1.3.11) \quad E[\{I(X_{Ni} \le u) - F_{Ni}(u)\}\{I(X_{Ni} \le v) - F_{Ni}(v)\}]$$

$$= F_{Ni}(u) \wedge F_{Ni}(v) - F_{Ni}(u) F_{Ni}(v)$$

$$= F_{Ni}(F_N^{-1}(t)) \wedge F_{Ni}(F_M^{-1}(t)) - F_{Ni}(F_N^{-1}(t)) F_{Ni}(F_M^{-1}(t)) .$$

By (i) and (3.11), substituting in (1.3.9) after taking expectations, we get (1.3.8) which completes the proof of Corollary (1.3.3).

1.4. The Limiting Distributions of the Processes $X_N(t)$ and $Y_N(t)$:

In section 3, we have found criteria for $\mathcal{L}_2$-convergence of the process $Y_N(t)$. Conditions for $X_N(t)$ are much simpler and are well-known.

We can rewrite $X_N(t)$ as,

(1.4.1) $$X_N(t) = \sqrt{N}\,(G_N(t) - t) \quad \text{where}$$

(1.4.2) $$G_N(t) = \frac{1}{N} \sum_{i=1}^{N} I(X_{Ni} \le F_N^{-1}(t))$$

$$E(X_N(s)X_N(t)) = K_N(s,t) =$$

$$= s \wedge t - \frac{1}{N} \sum_{i=1}^{N} F_{Ni}(F_N^{-1}(s))F_{Ni}(F_N^{-1}(t)) \quad .$$

Then the process $X_N(t)$ converges weakly to a Gaussian process $X(t)$ with $E(X(s)X(t)) = K(s,t) = \lim\limits_{N\to+\infty} K_N(s,t)$ if and only if the above limit exists.

Further, under the above condition

(1.4.2) $$\max_t |X_N(t) - X(t)| \longrightarrow 0 \quad \text{a.s.}$$

Hence $\mathcal{L}_2$-convergence is immediate. Proof of these can be found in Shorack [19].

Next we obtain conditions for convergence of the process $Y_N(t)$. The following theorem gives the conditions for $\mathcal{L}_2$-convergence of $Y_N(t)$.

Theorem 1.4.1 : Let conditions of theorem 1.3.2 (or corollary (1.3.3) be satisfied. In addition let

$$(1.4.3) \qquad (\max_{1\le i\le N} C_{Ni}) s_N^{-1} = O(\tfrac{1}{\sqrt{N}}) \ .$$

Then $Y_N(t) \xrightarrow{\mathcal{L}_2} Y(t)$ where $Y(t)$ is Gaussian with kernel

$$R(s,t) = \lim_{N\to+\infty} R_N(s,t) \ .$$

Proof : $\mathcal{L}_2$ convergence and convergence of the kernel have been established. It remains to show that the finite dimensional distributions of $Y_N(t)$ are asymptotically normal.

Let $t_1,t_2,\ldots,t_k \in T$ be arbitrary.

We have to show that the random vector $(Y_N(t_1),\ldots,Y_N(t_k))$ is asymptotically normal.

We use the Cramer-Wold criterion. Let $a_1,\ldots,a_k$ be arbitrary constants. It suffices to show that

$$\sum_{j=1}^{k} a_j\, Y_N(t_j) \quad \text{is asymptotically normal.} \tag{1.4.4}$$

Again to simplify our notation, we write,

$$F_N^{-1}(t_j) = u_j\ , \quad j = 1,2,\ldots,k\ .$$

Then

$$\sum_{j=1}^{k} a_j Y_N(t_j) = \sum_{j=1}^{k} a_j \sum_{i=1}^{N} \frac{C_{Ni}}{s_N}\{I(X_{Ni} \le u_j) - F_{Ni}(u_j)\}\ . \tag{1.4.5}$$

Set

$$\alpha_{Ni} = \frac{\sqrt{N}\ C_{Ni}}{s_N}\ ;\ \text{then}\quad \alpha_{Ni} = 0(1)\ ,\quad \text{and} \tag{1.4.6}$$

$$Y_N(u_j) = \alpha_{Ni}\{I(X_{Ni} \le u_j) - F_{Ni}(u_j)\}\ .$$

Then ,

$$\sum_{j=1}^{k} a_j Y_N(t_j) = \frac{1}{\sqrt{N}} \sum_{j=1}^{k} a_j \sum_{i=1}^{N} Y_{Ni}(u_j) \tag{1.4.7}$$

$$= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \{\sum_{j=1}^{k} a_j Y_{Ni}(u_j)\}\ .$$

Let $\sigma_{Ni}^2 = \operatorname{Var}\{\sum_{j=1}^{k} a_j Y_{Ni}(u_j)\}$

$$\sigma_N^2 = \sum_{i=1}^{N} \sigma_{Ni}^2\ . \tag{1.4.8}$$

Observe that because of (1.4.6), each term of the sum in (1.4.7), viz., $\{\sum_{j=1}^{k} a_j Y_{Ni}(u_j)\}$ is bounded uniformly in N. Further, in (1.4.8), σ_N^2 is a monotone increasing sequence since its terms are all nonnegative. Thus, if σ_N^2 is bounded above, it converges to some σ_0^2 and then $\sum_{j=1}^{k} a_j Y_N(t_j)$ is degenerate. (Because its variance is $\frac{1}{N}\sigma^2 \to 0$ if $\sigma_N^2 \to \sigma_o^2$, if σ_N^2 is bounded).

If on the other hand σ_N^2 is unbounded, $\sigma_N^2 \to +\infty$, we have asymptotic normality of $\sum_{j=1}^{k} a_j Y_N(t_j)$ by the Bounded Liapunov theorem.

Thus, in either case we have asymptotic normality (possibly degenerate) of $\sum_{j=1}^{k} a_j Y_N(t_j)$. Hence by the Cramer-Wold criterion, the process $Y_N(t)$ is asymptotically (possibly degerate) normal. Thus the proof of theorem 4.1 is complete.

1.5. Asymptotic Normality of Certain Stochastic Integrals of the Processes $X_N(t)$ and $Y_N(t)$.

In this section, we will obtain conditions for the asymptotic normality of the stochastic integrals,

(1.5.1) $$X_N = \frac{1}{\sqrt{N}\, s_N} \int_0^1 X_N(t)\varphi'(t)\,dc(t) \quad \text{and}$$

(1.5.2) $$Y_N = \int_0^1 Y_N(t)\varphi'(t)\,dt \quad \text{where}$$

$$(1.5.3) \qquad c(t) = \sum_{i=1}^{N} c_{Ni} F_{Ni}(F_N^{-1}(t)) \quad \text{and}$$

φ is a known function.

This in turn will enable us to obtain asymptotic normality of the simple linear rank statistic S_N defined in (1.1.1).

In the next theorem, we will obtain bounds for the variances of X_N and Y_N.

Variance Inequalities: Next theorem is crucial for obtaining the limiting normality of X_N and Y_N.

Theorem 1.5.1 : Let $X_{N1},\ldots,X_{NN}$ be independent random variables with continuous distribution functions $F_{N1},\ldots,F_{NN}$. Under the assumptions,

(a) $$\frac{\max_{1\le i\le N}|c_{Ni}|}{s_N} = O\left(\frac{1}{\sqrt{N}}\right)$$

(b) $|\varphi'(t)| \le K|t(1-t)|^{\delta-3/2}$, for some $\delta > 0$,

we have,

$$(1.5.4) \qquad \operatorname{Var} Y_N \le \frac{K^2}{\delta^2}\, 2^{5-4\delta}\, O(1)$$

$$(1.5.5) \qquad \operatorname{Var} X_N \le \frac{K^2}{\delta^2} 2^{5-4\delta} O(1)$$

<u>where</u> X_N <u>and</u> Y_N <u>are the random variables defined in (1.5.1) and (1.5.2) respectively, corresponding to the sequences</u> $(X_{N1},\ldots,X_{NN})$ <u>and</u> $(F_{N1},\ldots,F_{NN})$.

<u>Proof</u> : First consider Y_N .

$$Y_N = \int_0^1 Y_N(t)\varphi'(t)dt.$$

Clearly, $EY_N = 0$. Thus

$$(1.5.6) \qquad \operatorname{Var} Y_N = E\int_0^1\int_0^1 Y_N(s)Y_N(t)\varphi'(s)\varphi'(t)dsdt$$

$$= \int_0^1\int_0^1 R_N(s,t)\varphi'(s)\varphi'(t)dsdt \text{ , if it exists .}$$

By (1.3.5),

$$R_N(s,t) = \frac{1}{s_N^2}\sum_{i=1}^N C_{Ni}^2\{F_{Ni}(F_N^{-1}(s))(1-F_{Ni}(F_N^{-1}(t)))\} \quad s \le t \text{ .}$$

By the Cauchy-Schwarz inequality,

$$|R_N(s,t)| \le \{R_N(s,s)R_N(t,t)\}^{\frac{1}{2}} \text{ .}$$

Thus, substituting in (1.5.6) we get,

$$(1.5.7) \quad \operatorname{Var} Y_N \le \{\int_0^1 \{R_N(t,t)\}^{\frac{1}{2}} \varphi'(t)\,dt\}^2$$

$$= [\int_0^1 \{\sum_{i=1}^N \frac{C_{Ni}^2}{s_N^2} (F_{Ni}(F_N^{-1}(t))(1 - F_{Ni}(F_N^{-1}(t))\}^{\frac{1}{2}} \varphi'(t)\,dt]^2$$

Consider,

$$(1.5.7)' \quad R_N(t,t) = \sum_{i=1}^N \frac{C_{Ni}^2}{s_N^2} F_{Ni}(F_N^{-1}(t))(1 - F_{Ni}(F_N^{-1}(t)))$$

$$\le \frac{1}{N} \sum_{i=1}^N F_{Ni}(F_N^{-1}(t))(1 - F_{Ni}(F_N^{-1}(t)))\ 0(1) \quad .$$

Since, $0 \le F_{Ni}(F_N^{-1}(t)) \le 1$ and

$0 \le 1 - F_{Ni}(F_N^{-1}(t)) \le 1$ we have

$$(1.5.8) \quad R_N(t,t) \le 0(1)\ \frac{1}{N} \sum_{i=1}^N F_{Ni}(F_N^{-1}(t)) = t\ 0(1) \quad \text{and}$$

$$(1.5.9) \quad R_N(t,t) \le 0(1)\ \frac{1}{N} \sum_{i=1}^N (1 - F_{Ni}(F_N^{-1}(t))) = (1-t).\ 0(1) \ .$$

Thus,

$$(1.5.10) \quad \int_0^1 \{R_N(t,t)\}^{\frac{1}{2}} \varphi'(t)\,dt = \int_0^{\frac{1}{2}} \{R_N(t,t)\}^{\frac{1}{2}} \varphi'(t)\,dt$$

$$+ \int_{\frac{1}{2}}^1 \{R_N(t,t)\}^{\frac{1}{2}} \varphi'(t)\,dt.$$

Next, we have by (1.5.8) and assumption (b)

$$\left|\int_0^{\frac{1}{2}}\{R_N(t,t)\}^{\frac{1}{2}}\varphi'(t)\ dt\right| \le \int_0^{\frac{1}{2}} t^{\frac{1}{2}}|\varphi'(t)|\ dt \quad O(1)$$

$$\le K \int_0^{\frac{1}{2}} t^{\frac{1}{2}}\{t(1-t)\}^{\delta-3/2}\ dt \quad O(1)$$

$$\le K \int_0^{\frac{1}{2}} t^{\frac{1}{2}}\ t^{\delta-3/2}\ (1-\tfrac{1}{2})^{\delta-3/2}\ dt \quad O(1)$$

$$= K\frac{1}{2^{\delta-3/2}} \int_0^{\frac{1}{2}} t^{\delta-1}\ dt = \frac{k}{\delta}\ 2^{3/2-2\delta} \quad O(1) \quad .$$

Similarly by using (1.5.9) we get,

$$\int_{\frac{1}{2}}^1\{R_N(t,t)\}^{\frac{1}{2}}\varphi'(t)dt \le \frac{K}{\delta}\ 2^{3/2-\delta} \quad O(1)\ .$$

Thus, we get

$$\text{(1.5.11)} \quad \left|\int_0^1\{R_N(t,t)\}^{\frac{1}{2}}\varphi'(t)dt\right| \le O(1)\ \frac{2K}{\delta}\ 2^{3/2-2\delta} = \frac{K}{\delta}\ 2^{5/2-2\delta}\ O(1)\ .$$

Substituting in (1.5.9), we get

$$\left\{\int_0^1\{R_N(t,t)\}^{\frac{1}{2}}\varphi'(t)dt\right\}^2 \le O(1)\ \frac{K^2}{\delta^2}\ 2^{5-4\delta} \quad .$$

This proves (1.5.4).

Next consider

$$X_N = \frac{1}{s_N\sqrt{N}} \int_0^1 X_N(t)\varphi'(t)dc(t)\ .$$

Without loss of generality, assume $C_{Ni} \ge 0$; otherwise

we replace c_{Ni} by $|c_{Ni}|$

$$\text{Var } X_N = \frac{1}{Ns_N^2} \int_0^1 \int_0^1 K_N(s,t)\varphi'(s)\varphi'(t)dc(s)dc(t)$$

$$\le \frac{1}{Ns_N^2} \int_0^1 \int_0^1 |K_N(s,t)||\varphi'(s)\varphi'(t)|dc(s)dc(t)$$

$$\le \frac{1}{\sqrt{N}s_N} \int_0^1 \{|K_N(t,t))\}^{\frac{1}{2}}|\varphi'(t))dc(t)]^2 \quad .$$

Observe that $\dfrac{\max|c_{Ni}|}{\sqrt{N}\, s_N} = O(\frac{1}{N})$. Thus

$$\text{Var } X_N \le O(1)\{\int_0^1\{K_N(t,t)\}^{\frac{1}{2}}|\varphi'(t)|dt\}^2$$

$$= \{\int_0^1\{\frac{1}{N}\sum_{i=1}^N F_{Ni}(F_N^{-1}(t))(1 - F_{Ni}(F_N^{-1}(t)))\}^{\frac{1}{2}}|\varphi'(t)|dt\}^2 \quad O(1) \; .$$

This we have already estimated and found to be

$$\le \frac{K^2}{\delta^2} \; 2^{5-4\delta} \quad O(1) \quad .$$

Thus $\text{Var } X_N \le O(1) \frac{K^2}{\delta^2} 2^{5-4\delta}$, which completes the proof.

We are now in a position to prove the following theorems on the asymptotic normality of X_N and Y_N .

<u>Theorem 1.5.2</u> : <u>Let</u> $X_N(t)$ <u>and</u> X_N <u>be defined as</u> before. Assume

(a) $E\, X_N(s)X_N(t) = K_N(s,t) \to K(s,t)$, $\forall s,t \in T$

(b) $\max_{1 \le i \le N} |c_{Ni}|/s_N = 0(\frac{1}{\sqrt{N}})$

(c) $|\varphi'(t)| \le K|t(1-t)|^{\delta-3/2}$, for some $\delta > 0$.

Then given any $\varepsilon > 0$, *there exist* N_0 , *such that* $N \ge N_0$ *entails,* $\|X_N - X\| < \varepsilon$, *where*

(1.5.12) $X = \frac{1}{s_N\sqrt{N}} \int_0^1 X(t)\ \varphi'(t)dc(t)$ *is a Gaussian random variable with mean* 0 *and variance*

$$\|X\|^2 = \frac{1}{Ns_N^2} \int_0^1 \int_0^1 K(s,t)\varphi'(s)\varphi'(t)dc(s)dc(t)$$

where, as before

$$c(t) = \sum_{i=1}^{N} c_{Ni}F_{Ni}(F_N^{-1}(t)) \ .$$

Proof : First we observe that convergence of $K_N(s,t)$ to $K(s,t)$ entails,

$$\max_{0 \le t \le 1} |X_N(t) - X(t)| \longrightarrow 0 \text{ by condition (1.4.2) .}$$

Thus $\|X_N(t) - X(t)\| \longrightarrow 0$ and $X(t)$ is Gaussian with mean 0 and kernel $K(s,t) = \lim_{N\to+\infty} K_N(s,t)$.

Next, we observe that, assuming $c_{Ni} \ge 0$, w.l.o.g.,

$$(1.5.13)\quad \|X\| = \frac{1}{\sqrt{N}\, s_N}\left\{\int_0^1 \int_0^1 K(s,t)\varphi'(s)\varphi'(t)\,dc(v)\,dc(t)\right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{N}\, s_N} \int_0^1 \{K(t,t)\}^{\frac{1}{2}}|\varphi'(t)|\,dc(t)$$

$$\leq \lim_N \sup \frac{1}{\sqrt{N}\, s_N} \int_0^1 K_N(t,t)|\varphi'(t)|\,dc(t) < +\infty$$

by theorem (1.5.1) .

Next,

$$\|X_N - X\| = \left\| \frac{1}{\sqrt{N}\, s_N} \int_0^1 (X_N(t) - X(t))\varphi'(t)\,dc(t) \right\|$$

$$\leq \int_0^1 \frac{1}{\sqrt{N}\, s_N} \|X_N(t) - X(t)\|\,|\varphi'(t)|\,dc(t)$$

$$< +\infty \; .$$

Thus we can use the generalized dominated convergence theorem, as $\|X_N(t) - X(t)\| \to 0$, we must have,

$$\|X_N - X\| \to 0 \; .$$

Thus for N sufficiently large,

$$\|X_N - X\| < \epsilon$$

which completes the proof.

<u>Theorem 1.5.3</u> : <u>Let in addition to the conditions (b) and (c) of theorem 1.5.2 above the conditions of either theorem 1.3.2 or of Corollary 1.3.3 be satisfied. Then, given any</u> $\varepsilon > 0$, <u>there exists</u> N_0 <u>such that</u> $N \geq N_0$ <u>entails</u>

$$\|Y_N - Y\| < \varepsilon \qquad \underline{\text{where}}$$

$$Y = \int_0^1 Y(t)\varphi'(t)dt \quad \underline{\text{and}} \tag{1.5.14}$$

$Y(t)$ <u>is a Gaussian process with mean</u> 0 <u>and kernel</u>

$$\begin{aligned} R(s,t) &= \lim_{N\to+\infty} R_N(s,t) \\ &= \lim_{N\to+\infty} \frac{1}{s_N^2} \sum_{i=1}^{N} c_{Ni}^2 F_{Ni}(F_N^{-1}(s))(1-F_N^{-1}(t))) \end{aligned} \tag{1.5.15}$$

<u>with</u> $s \leq t$.

<u>Proof</u> : By theorem 1.4.1, we have $\mathcal{L}_2$-convergence of Y $Y_N(t)$ to a Gaussian process $Y(t)$.

Again observe that

$$\begin{aligned} \|Y\| &= \{\int_0^1 \int_0^1 R(s,t)\varphi'(s)\varphi'(t)dsdt\}^{\frac{1}{2}} \\ &\leq \int_0^1 \{R(t,t)\}^{\frac{1}{2}}|\varphi'(t)|dt \\ &\leq \lim_N \sup \int_0^1 \{R_N(t,t)\}^{\frac{1}{2}}|\varphi'(t)|dt \quad < +\infty \end{aligned}$$

by theorem 1.5.1.

Thus

$$\|Y_N - Y\| = \|\int_0^1 (Y_N(t) - Y(t))\varphi'(t)dt\|$$

$$\le \int_0^1 \|Y_N(t) - Y(t)\| |\varphi'(t)| dt$$

$$\longrightarrow 0 \quad \text{as} \quad N \to +\infty .$$

Since $\|Y_N(t) - Y(t)\| |\varphi'(t)| \to 0$ and the integral is dominated by

$$\int_0^1 (\|Y_N(t)\| + \|Y(t)\|) |\varphi'(t)| dt < +\infty$$

Thus $\|Y_N - y\| < \epsilon$ for N large enough. This proves the theorem.

1.6. The Limiting Distribution of a Simple Linear Rank Statistic:

We are now in a position to prove the asymptotic normality of the statistic defined in (1.1.1)

Recall that

$$S_N = \sum_{i=1}^{N} C_{Ni} \varphi \left(\frac{R_{Ni}}{N+1}\right) .$$

We make the following assumptions:

(a) $|\varphi(t)| \le K\{t(1-t)\}^{\delta-\frac{1}{2}}$, some $\delta > 0$

$|\varphi'(t)| \le K\{t(1-t)\}^{\delta-3/2}$ K being a generic constant.

(b) Conditions of theorem 1.3.2 (or Corollary 1.3.3) are satisfied and the kernel $K_N(s,t) \to K(s,t)$.

(c) $(\max_{1 \le i \le N} C_{Ni})(\text{Var } S_N)^{-\frac{1}{2}} = 0(\frac{1}{\sqrt{N}})$.

We then have the following theorem:

<u>Theorem 1.6.1</u> : <u>Let the independent random variables</u> $X_{N1},\dots,X_{NN}$ <u>with continuous distribution functions</u> $F_{N1},\dots,F_{NN}$ <u>be such that conditions</u> (a) , (b) <u>and</u> (c) <u>are satisfied</u>.

<u>Then,</u>

(1.6.1) $$\mathcal{L}\left(\frac{S_N - \mu_N}{(\text{Var } S_N)^{\frac{1}{2}}}\right) \longrightarrow \mathcal{N}(0,1)$$

$$\text{Var } S_N \ne 0$$

where

(1.6.2) $$\mu_N = \int_0^1 \varphi(t)\, dc(t)$$

$$c(t) = \sum_{i=1}^{N} C_{Ni} F_{Ni}(F_N^{-1}(t))$$

and Var S_N can be taken to be

$$(1.6.3)\qquad s_N^2 = \sum_{i=1}^{N} \operatorname{Var} A_{Ni}(X_{Ni}) \quad \text{where}$$

$$(1.6.4)\qquad A_{Ni}(x) = \frac{1}{N}\sum_{j=1}^{N} (C_{Nj} - C_{Ni}) \int_{-\infty}^{\infty} \{I(x \le y) - F_{Ni}(y)\} \varphi'(F_N(y))\, dF_{Nj}(y)$$

Remark : The condition on φ is the same as that imposed by Puri and Sen [22] and slightly stronger than Hàjek [12] . Otherwise the theorem is not different from Hàjeck's. The conclusion of the theorem is slightly stronger in the sense we have obtained the centering constant μ_N not given by Hàjek. This μ_N arises naturally during the course of the proof without a separate analysis as done by Hoeffding [14] . Hàjek's principal tool is his remarkable but difficult variance inequality which is replaced here by the much simpler theorem 1.5.1.

Proof of Theorem 1.6.1 : We can write the statistic S_N as

$$(1.6.5)\qquad S_N = \int_0^1 \varphi\left(\frac{N}{N+1} H_N(t)\right) dC_N(t) \quad \text{where}$$

$$(1.6.6)\qquad C_N(t) = \sum_{i=1}^{N} C_{Ni} I(X_{Ni} \le F_N^{-1}(t))$$

$$(1.6.7)\qquad H_N(t) = \frac{1}{N}\sum_{i=1}^{N} I(X_{Ni} \le F_N^{-1}(t)) .$$

It is easy to check that

(1.6.8) $S_N = \mu_N + B_{1N} + B_{2N} + D_{1N} + D_{2N} + D_{3N}$ where

μ_N has been defined in (6.2)

(1.6.9) $B_{1N} = \int_0^1 \varphi(t)d(C_N(t) - C(t))$

(1.6.10) $B_{2N} = \int_0^1 (H_N(t) - t)\varphi'(t)dc(t)$

(1.6.11) $D_{1N} = \frac{-1}{N+1} \int_0^1 H_N(t)\varphi'(t)\,dc(t)$

(1.6.12) $D_{2N} = \int_0^1 (H_N(t) - t)\varphi'(t)d(C_N(t)-C(t))$

(1.6.13) $D_{3N} = \int_0^1 \{\varphi(\frac{N}{N+1} t) - \varphi(t) - (\frac{N}{N+1} H_N(t)-t)\varphi'(t)\}dC_N(t)$.

We will show that (i) $|\mu_N| < +\infty$

(ii) $\frac{1}{s_N}(B_{1N} + B_{2N})$ is asymptotically normal

(iii) $D_{iN} = o_p(s_N)$, $i = 1,2,3$.

The proof of (i) and (iii) are given in the next chapter. We prove (ii) below.

Integrating B_{1N} by parts, we have

$$B_{1N} = -\int (C_N(t) - C(t))\varphi'(t)dt + (C_N(t) - C(t))\varphi(t)\Big|_0^1 .$$

(1.6.14) Let $Z_N(t) = \varphi(t)\{C_N(t) - C(t)\}$.

<u>Claim</u> : $Z_N(t) = o_p(s_N)$ as $t \to 0$ or 1

$$\frac{\text{Var } Z_N(t)}{s_N^2} = \frac{\varphi^2(t)}{s_N^2} \sum_{i=1}^{N} C_{Ni}^2 \{F_{Ni}(F_N^{-1}(t))(1-F_{Ni}(F_N^{-1}(t)))\}$$

$$\le O(1)\ \varphi^2(t)\ \frac{1}{N} \sum_{i=1}^{N} F_{Ni}(F_N^{-1}(t))(1-F_{Ni}(F_N^{-1}(t)) .$$

Observe that $0 \le F_{Ni}(F_N^{-1}(t)) \le 1$ and

$$0 \le 1 - F_{Ni}(F_N^{-1}(t)) \le 1 .$$

Hence their product is less than the individual factors.

Hence

$$\frac{\text{Var } Z_N(t)}{s_N^2} \le O(1)\ \varphi^2(t)\ t \quad \text{and}$$

$$\frac{\text{Var } Z_N(t)}{s_N^2} \le O(1)\ \varphi^2(t)(1-t) \ , \forall t .$$

Consequently,

$$\frac{\text{Var } Z_N(t)}{s_N^2} \le O(1)\ K^2\{t(1-t)\}^{2\delta-1}\ t \to 0 \quad \text{as} \quad t \to 0$$

and $$\frac{\text{Var } Z_N(t)}{s_N^2} \le O(1)\ k^2\{t(1-t)\}^{2\delta-1}(1-t) \to 0 \quad \text{as} \quad t \to 1 .$$

Thus $\text{Var}\left(\frac{Z_N(t)}{s_N}\right) \to 0$ as $t \to 0$ or 1

$$\Rightarrow Z_N(t) = o_p(s_N) \quad \text{as} \quad t \to 0 \quad \text{or} \quad 1 .$$

Next we consider

$$\frac{1}{s_N} (B_{1N} + B_{2N}) \quad .$$

Observe that $\dfrac{B_{1N}}{s_N} = -Y_N$ and

$$\frac{B_{2N}}{s_N} = X_N$$

where X_N and Y_N are as defined in (1.5.1) and (1.5.2).

By theorems 1.5.1 and 1.5.2, there exist Gaussian random variables, X and Y such that for N sufficiently large,

$$\|X_N - X\| < \varepsilon/2 \quad \text{and} \quad \|Y_N - Y\| < \varepsilon/2 \ .$$

Thus $\|(X_N - Y_N) - (X - Y)\| < \varepsilon$. But $X - Y$ is Gaussian. This proves the theorem.

Variance computation : We can now find an expression for s_N^2 .

Letting $Y = F_N^{-1}(t)$ and observing that

$$C(t) = \sum_{j=1}^{N} C_{Nj} \, F_{Nj}(F_N^{-1}(t)) = \sum_{j=1}^{N} C_{Nj} F_{Nj}(Y) = C(y) \ , \text{ say.}$$

We can write,

(1.6.14) $$B_{1N} + B_{2N} = \int_{-\infty}^{\infty} \frac{1}{N} \sum_{i=1}^{N} \{I(X_{Ni} \le y) - F_{Ni}(y)\} \varphi'(F_N(y))\, dc(y)$$

$$- \int_{-\infty}^{\infty} \sum_{i=1}^{N} C_{Ni}\{I(X_{Ni} \le y) - F_{Ni}(y)\} \varphi'(F_N(y))\, dF_N(y)$$

$$= \sum_{i=1}^{N} A_{Ni}(X_{Ni}) \quad \text{where}$$

$$A_{Ni}(x) = \frac{1}{N} \int_{-\infty}^{\infty} \{I(x \le y) - F_{Ni}(y)\} \varphi'(F_N(y))\, dc(y)$$

$$- C_{Ni} \int_{-\infty}^{\infty} \{I(x \le y) - F_{Ni}(y))\} \varphi'(F_N(y))\, dF_N(y)$$

$$= \frac{1}{N} \sum_{j=1}^{N} C_{Nj} \int_{-\infty}^{\infty} \{I(x \le y) - F_{Ni}(y)\} \varphi'(F_N(y))\, dF_{Nj}(y)$$

$$- \frac{1}{N} \sum_{j=1}^{N} C_{Ni}\{I(x \le y) - F_{Ni}(y)\} \varphi'(F_N(y))\, dF_{Nj}(y) .$$

Thus we have,

(1.6.15) $$A_{Ni}(x) = \frac{1}{N} \sum_{j=1}^{N} (C_{Nj} - C_{Ni}) \int_{-\infty}^{\infty} \{I(x \le y) - F_{Ni}(y)\} \varphi'(F_N(y))\, dF_{Nj}(y)$$

If $s_{Ni}^2 = \text{Var } A_{Ni}(X_{Ni})$, since X_{Ni} are all independent, we have

$$s_N^2 = \sum_{i=1}^{N} s_{Ni}^2 .$$

The remainder terms are all $o_p(s_N)$. Hence Var s_N

can be replaced by s_N^2 .

1.7. APPENDIX TO CHAPTER I :

On the Identification of the Hilbert Space spanned by the process $Y_N(t)$:

Although the material covered in this section is not essential to the material covered in any of the chapters, it provides some insight into the second order behaviour of the weighted empirical process $Y_N(t)$. We will explicitly identify the functions belonging to the Reproducing Kernel Hilbert Space $\mathcal{H}_N$ spanned by the process $Y_N(t)$.

In order to simplify the notation, we will drop the normalizing constant s_N and write

$$(1.7.1) \qquad Y_N(t) = \sum_{i=1}^{N} C_{Ni}\{I(X_{Ni} \le F_N^{-1}(t)) - F_{Ni}(F_N^{-1}(t))\} \quad \text{and}$$

$$R_N(s,t) = \sum_{i=1}^{N} C_{Ni}^2\{F_{Ni}(F_N^{-1}(s)) \wedge F_{Ni}(F_N^{-1}(t)) - F_{Ni}(F_N^{-1}(s))F_{Ni}(F_N^{-1}(t))\}$$

We identify the space $\mathcal{L}_2(Y_N)$ spanned by $Y_N(t)$ with a Hilbert space $\mathcal{H}_N$ of functions on T

$$\mathcal{H}_N \simeq \mathcal{L}_2(Y_N)$$

as follows :

For the stochastic integral

$$Y_g = \int_T Y_N(t)\,dg(t)$$

$$\Lambda : \mathcal{L}_2(Y_N) \longrightarrow \mathcal{H}_N$$

$$Y_g \longrightarrow f_g$$

$$f_g(s) = (f_g, R_{N,s}) = (\Lambda Y_g , \Lambda Y_N(s))$$

$$= (Y_g, Y_N(s)) = E(Y_g Y_N(s))$$

$$= E\{Y_N(s) \int_T Y_N(t)\,dg(t)\} = \int_T R_N(s,t)\,dg(t) \ .$$

Thus

(1.7.2) $$f_g(s) = (\Lambda Y_g)(s) = \int_T R_N(s,t)\,dg(t)$$

$$= \sum_{i=1}^{N} C_{Ni}^2 \int_T \{F_{Ni}(F_N^{-1}(s)) \wedge F_{Ni}(F_N^{-1}(t)) - F_{Ni}(F_N^{-1}(s))F_{Ni}(F_N^{-1}(t))\}\,dg(t) \ .$$

(1.7.3) $$\|f_g\|^2 = \|\Lambda Y_g\|^2 = \|Y_g\|^2 = \int_T \int_T R_N(s,t)\,dg(s)\,dg(t)$$

if it exists.

Thus $\mathcal{H}_N$ consists of all the functions $f_g(s) = \int_T R_N(s,t)\,dg(t)$ such that (7.3) is finite, (perhaps improper).

If $Y_g = \int_T Y_N(t)\,dg(t)$, $Y_h = \int_T Y_N(t)\,dh(t)$

$$\Lambda Y_g = f_g \ , \quad \Lambda Y_h = f_h$$

$$(1.7.4) \qquad (f_g, f_h) = (\Lambda Y_g , \Lambda Y_h) = (Y_g , Y_h)$$

$$= E\{\int_T Y_N(t)\,dg(t) \int_T Y_N(s)\,dh(s)\} = \int_T \int_T R_N(s,t)\,dg(s)\,dh(t)$$

$$= \sum_{i=1}^{N} C_{Ni}^2 \int_T \int_T \{F_{Ni}(F_N^{-1}(s)) \wedge F_{Ni}(F_N^{-1}(t)) - F_{Ni}(F_N^{-1}(s))F_{Ni}(F_N^{-1}(t))\}$$

$$dg(s)\,dg(t) \quad .$$

<u>Example</u> : Let $F_{N1} = \ldots = F_{NN} = F$ rectangular $(0,1)$. Then $R_N(s,t) = (s \wedge t - st) \sum_{i=1}^{N} C_{Ni}^2$. Let $g(t) = \log(t)$. Then

$$f_g(s) = \sum_{i=1}^{N} C_{Ni}^2 \int_0^1 (s \wedge t - st)\,dg(t) \quad .$$

It is clearly an improper integral. We will find the $\|f_g\|^2 = \int_0^1 \int_0^1 R_N(s,t)\,dg(t)\,dg(s)$.

$$\int_0^1 (s \wedge t - st)\,d \log t = \int_{[s<t]} s(1-t)\frac{1}{t}\,dt + \int_{[s\geq t]} t(1-s)\frac{1}{t}\,dt$$

$= -s \log (s)$. Thus

$$f_g(s) = -\sum_{i-1}^{N} C_{Ni}^2 \, s \log s \ .$$

$$\|f_g\| = \{-\sum_{i=1}^{N} C_{Ni}^2 \int_0^1 s \log s \ d \log s\}^{\frac{1}{2}} = \{\sum_{i=1}^{N} C_{Ni}^2\}^{\frac{1}{2}} \ .$$

Thus, although g is not of bounded variation on T,

$$Y_g = \int_0^1 Y_N(t)\,dg(t) \quad \text{exists.}$$

$$(\Lambda Y_g)(s) = f_g(s) = -\sum_{i=1}^{N} C_{Ni}^2 s \log s \ .$$

It is clear that a knowledge of f_g gives us no in information about the distribution of Y_g.

On the other hand if it were known that $Y_N(t)$ is Gaussian with mean 0 and kernel $R_N(s,t)$ above, f_g would completely determine the random variable Y_{g_N}, which would now be Gaussian with mean 0 and variance $\sum_{i=1} C_{Ni}^2$.

CHAPTER II

A Stability Approach to the Limiting Distribution of a Simple Linear Rank Statistic :

Preliminaries : In this chapter, a different proof of theorem 1.6.1 of Chapter I will be given using stability theorems from classical Probability Theory. The proof will be along the lines of the classical proof of the Chernoff-Savage theorem. Also the negligibility of the terms D_{iN} will be shown.

In order to conform to the standard terminology, we will use $H(x)$ in place of $F_N(x)$ used in the preceding chapter. That is to say,

$$H(x) = \frac{1}{N} \sum_{i=1}^{N} F_{Ni}(x) \quad .$$

We follow the proof given in Puri and Sen [22] for the Chernoff-Savage theorem, with some modifications necessiated by the greater generality of the problem.

The Limiting Distribution of a Simple Linear Linear Rank Statistic Under Alternatives :

2.1. **Notation and Terminology** : Let $\{X_{Ni} : 1 \le i \le N \, , \, N \ge 1\}$ be a sequence of independent random variables with continuous distribution functions,

$$F_{Ni}(x) = P(X_{Ni} \le x) \ .$$

Let $R_{N1}, R_{N2}, \ldots, R_{NN}$ be the corresponding ranks.

We define a simple linear rank statistic

$$S_N = \sum_{i=1}^{N} \varphi\left(\frac{R_{Ni}}{N+1}\right) C_{Ni} \qquad \text{where} \tag{2.1.1}$$

$\varphi : (0,1) \to \mathbb{R}$ is a known function and $C_{N1}, \ldots, C_{NN}$ are known regression constants.

We are interested in the limiting distribution of S_N .

<u>Assumptions</u> :

(a) $|\varphi(u)| \le K|u(1-u)|^{\delta-\frac{1}{2}}$,

$|\varphi'(u)| \le K|u(1-u)|^{\delta-3/2}$, for some $\delta > 0$

(b) If $s_N^2 = \text{Var } S_N$, then

$$\frac{\max\limits_{1\le i\le N} |C_{Ni}|}{s_N} = O(N^{-\frac{1}{2}}) \ .$$

<u>Note</u> : Throughout, K will indicate a generic constant.

We define

$$C_N(x) = \sum_{i=1}^{N} C_{Ni} I(X_{Ni} \le x) \tag{2.1.2}$$

$$(2.1.3) \qquad C(x) = \sum_{i=1}^{N} C_{Ni} F_{Ni}(x)$$

$$(2.1.4) \qquad H_N(x) = \frac{1}{N} \sum_{i=1}^{N} I(X_{Ni} \le x)$$

$$(2.1.5) \qquad H(x) = \frac{1}{N} \sum_{i=1}^{N} F_{Ni}(x) \quad .$$

Then the following inequalities are obvious:

$$(2.1.6) \qquad |C_N(x)| \le N \max_{1 \le i \le N} |C_{Ni}| H_N(x)$$

$$(2.1.7) \qquad |C(x)| \le N \max_{1 \le i \le N} |C_{Ni}| H(x)$$

2.2. Main Results:

Theorem 2.2.1 : Let S_N be as defined in (2.1.1).
Then $\mathcal{L}\left(\frac{S_N - \mu_N}{s_N}\right) \longrightarrow n(0,1)$

where $s_N^2 \ne 0$.

$$(2.2.1) \qquad \mu_N = \int_{-\infty}^{\infty} \varphi(H(x))\, dc(x) .$$

Also s_N^2 can be taken to be

$$(2.2.2) \qquad \sum_{i=1}^{N} \text{Var } A_{Ni}(X_{Ni})$$

where

$$(2.2.3) \quad A_{Ni}(x) = \frac{1}{N} \sum_{j=1}^{N} (C_{Ni} - C_{Nj}) \int_{x_0}^{x} \varphi'(H(y))\, dF_{Nj}(y) \; ; \quad x_0 \text{ arbitrary} .$$

<u>Proof</u> : Proof will be along the lines of Puri and Sen [22] .

We can write

$$(2.2.4) \qquad S_N = \int_{-\infty}^{\infty} \varphi\left(\frac{N}{N+1} H_N(x)\right) dC_N(x)$$

$$= \mu_N + B_{1N} + B_{2N} + D_{1N} + D_{2N} + D_{3N} \qquad \text{where}$$

μ_N as in (2.2.2)

$$(2.2.5) \qquad B_{1N} = \int_{-\infty}^{\infty} \varphi(H(x))\, d(C_N(x) - C(x))$$

$$(2.2.6) \qquad B_{2N} = \int_{-\infty}^{+\infty} (H_N(x) - H(x))\, \varphi'(H(x))\, dc(x)$$

$$(2.2.7) \qquad D_{1N} = -\frac{1}{N+1} \int_{-\infty}^{\infty} H_N(x)\, \varphi'(H(x))\, dC_N(x)$$

$$(2.2.8) \qquad D_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x))\, \varphi'(H(x))\, d(C_N(x) - C(x))$$

$$(2.2.9) \qquad D_{3N} = \int_{-\infty}^{\infty} \left\{ \left[\varphi\left(\frac{N}{N+1} H_N(x)\right) - \varphi(H(x))\right] - \left(\frac{N}{N+1} H_N(x) - H(x)\right) \varphi'(H(x)) \right\} dC_N(x) .$$

The proof will be accomplished if we show the following:

(i) $|\mu_N| < +\infty$

(ii) $\frac{1}{s_N}(B_{1N}+B_{2N})$ is a symptotically normal.

(iii) $D_{iN} = o_p(s_N)$, $i = 1,2,3$.

<u>Proof of (i)</u> : $|\mu_N| = |\int_{-\infty}^{\infty} \varphi(H(x))dC(x)|$

$\leq N \max_{1\leq i\leq N} |C_{Ni}| \int_{-\infty}^{\infty} |\varphi(H(x))|dH(x)$ by (1.7) .

Hence by assumption (a)

(2.2.10) $|\mu_N| \leq N \max_{1\leq i\leq N} |C_{Ni}| K \int_0^1 \{u(1-u)\}^{\delta-\frac{1}{2}} du < +\infty$.

<u>Proof of (ii)</u> : To show that $\frac{1}{s_N}(B_{1N}+B_{2N})$ is asymptotically normal.

We will verify that B_{1N} and B_{2N} satisfy the Liapunov condition and then apply the C_r-inequality

(2.2.10) $B_{1N} = \int_{-\infty}^{\infty} \varphi(H(x))d(C_N(x)-C(x))$

$= \sum_{i=1}^{N} \{B(X_{Ni}) - EB(X_{Ni})\}C_{Ni}$ where

$B(X_{Ni}) = \varphi(H(X_{Ni}))$.

Consider next,

$$(2.2.12)\qquad \sum_{i=1}^{N} E|C_{Ni}B(X_{Ni})|^{2+\delta'} \le \max_{1\le i\le N} |C_{Ni}|^{2+\delta'} \sum_{i=1}^{N} E|B(X_{Ni})|^{2+\delta'}$$

$$= \max_{1\le i\le N} |C_{Ni}|^{2+\delta'} \sum_{i=1}^{N} \int_{-\infty}^{\infty} |\varphi(H(x))|^{2+\delta'} dF_{Nl}(x) \ .$$

Choose δ' s.t. $(2+\delta')(\delta-\frac{1}{2}) > -1$. Then in (2.2.12) we have,

$$(2.2.13)\quad \sum_{i=1}^{N} E|C_{Ni}B(X_{Ni})|^{2+\delta'} \le \max_{1\le i\le N} |C_{Ni}|^{2+\delta'} \ .$$

$$\cdot\ N \cdot \frac{1}{N} \sum_{i=1}^{N} \int |\varphi(H(x))|^{2+\delta'} dF_{Ni}(x)$$

$$\le \max_{1\le i\le N} |C_{Ni}|^{2+\delta'} \cdot N\ K \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{(2+\delta')(\frac{1}{2}-\delta)} dH(x)$$

$$\le K'\ N \max |C_{Ni}|^{2+\delta'} \ ; \text{ where}$$

$$K' = K \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{(2+\delta')(\frac{1}{2}-\delta)} dH(x) < +\infty \ .$$

Also by Jensen's inequality,

$$|EB(X_{Ni})C_{Ni}|^{2+\delta'} \le |C_{Ni}|^{2+\delta'} E|B(X_{Ni})|^{2+\delta'} \ .$$

$$(2.2.14)\quad \text{Hence } E|\{B(X_{Ni}) - EB(X_{Ni})\}C_{Ni}]^{2+\delta'}$$

$$\le 2^{2+\delta'} E|B(X_{Ni})|^{2+\delta'} |C_{Ni}|^{2+\delta'} \ .$$

By (2.2.13) and (2.2.14) in (2.2.11), we have

$$(2.2.15)\qquad \sum_{i=1}^{N} E|C_{Ni}\{B(X_{Ni}) - EB(X_{Ni})\}|^{2+\delta'}$$

$$\le 2^{2+\delta'}\, N \cdot \max_{1\le i\le N} |C_{Ni}|^{2+\delta'} K' \ .$$

Hence,

$$(2.2.16)\qquad \frac{1}{s_N^{2+\delta'}} \sum_{i=1}^{N} E|C_{Ni}\{B(X_{Ni}) - E\, B(X_{Ni})\}|^{2+\delta'}$$

$$\le K\, N \cdot \max_{1\le i\le N} \frac{|C_{Ni}|^{2+\delta'}}{s_N^{2+\delta'}} = N\, O\Big(\frac{1}{N^{1+\delta'/2}}\Big)$$

$$= O\Big(\frac{1}{N^{\delta'/2}}\Big) \to 0 \quad \text{as} \quad N \to +\infty \ .$$

Hence the Liapunov condition is satisfied.

Now consider,

$$B_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x))\varphi'(H(x))dc(x) \ .$$

Integrating by parts,

$$(2.2.17)\qquad B_{2N} = B^*(x)\{H_N(x) - H(X)\}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} B^*(x)d(H_N(x) - H(x))$$

where,

$$(2.2.18)\qquad B^*(x) = \int_{x_0}^{x} \varphi'(H(x))dc(x) \ , \ x_0 \text{ arbitrary s.t. } H(x_0)>0 \ .$$

We will later show that the first term in (2.2.17) is

$o_p(s_N)$. Consider,

$$\int_{-\infty}^{\infty} B^*(x)\,d(H_N(x)-H(x))) = \sum_{i=1}^{N} \frac{B^*(X_{Ni})-E\,B^*(X_{Ni})}{N} \tag{2.2.19}$$

We will again verify the Liapunov condition,

$$\frac{1}{s_N^{2+\delta'}} \sum_{i=1}^{N} E\left|\frac{B^*(X_{Ni})-E\,B(X_{Ni})}{N}\right|^{2+\delta'} \longrightarrow 0 \ , \quad \text{as} \quad N \to +\infty \ .$$

Clearly, as before it suffices to consider

$$\frac{1}{s_N^{2+\delta'}} \sum_{i=1}^{N} E\left|\frac{B^*(X_{Ni})}{N}\right|^{2+\delta'} = \frac{1}{s_N^{2+\delta'}N^{2+\delta'}} \sum_{i=1}^{N} E|B^*(X_{Ni})|^{2+\delta'} \tag{2.2.20}$$

$$= \frac{1}{N^{2+\delta'}(s_N)^{2+\delta'}} \sum_{i=1}^{N} \int_{-\infty}^{\infty} \left|\int_{x_0}^{x} \varphi'(H(y))\,dH(y)\right|^{2+\delta'} dF_{Ni}(x)$$

$$\le \frac{1}{s_N^{2+\delta'}} \ \frac{1}{N^{2+\delta'}} \sum_{i=1}^{N} N^{2+\delta'} \int_{-\infty}^{\infty} \max_{1\le i\le N} |C_{Ni}|^{2+\delta'} \left|\int_{x_0}^{x} \varphi'(H(y))\,dH(y)\right|^{2+\delta'} dF_{Ni}(x)$$

$$\le \frac{\max|C_{Ni}|^{2+\delta'}}{s_N^{2+\delta'}} \sum_{i=1}^{N} \int_{-\infty}^{\infty} |\varphi(H(x)) + \varphi(H(x_0))|^{2+\delta'} dF_{Ni}(x)$$

$$= 0\left(\frac{1}{N^{\delta'/2}}\right) \frac{1}{N} \sum_{i=1}^{N} \int_{-\infty}^{\infty} |\varphi(H(x)) + \varphi(H(x_0))|^{2+\delta'} dF_{Ni}(x)$$

$$= 0\left(\frac{1}{N^{\delta'/2}}\right) \int_{-\infty}^{\infty} |\varphi(H(x)) + \varphi(H(x_0))|^{2+\delta'} d\,H(x)$$

$$\longrightarrow 0 \quad \text{as} \quad N \to +\infty \ .$$

Since $(2+\delta')(\delta-\tfrac{1}{2}) > -1$ and $\varphi(H(x_0))$ is a constant.

Again,

$$(2.2.21) \quad \frac{1}{s_N^{2+\delta'}} \sum_{i=1}^{N} \left| \frac{B^*(X_{Ni}) - E\, B^*(X_{Ni})}{N} \right|^{2+\delta'} \longrightarrow 0 \ .$$

By (2.2.16) and (2.2.21), the C_r-inequality, $B_{1N} + B_{2N}$ satisfies the Liapunov condition.

In (2.2.17), consider $B^*(x)\{H_N(x) - H(x)\}\Big|_{-\infty}^{\infty} = \beta(x)\Big|_{\infty}^{\infty}$

$$\left|\frac{\beta(x)}{s_N}\right| = |[H_N(x) - H(x)]|\sqrt{N}\, \frac{1}{s_N\sqrt{N}}\, \left|\int_{x_0}^{x} \varphi'(H(y))\,dc(y)\right|$$

$$\le \sqrt{N}\, |H_N(x) - H(x)|\, \left|\int_{x_0}^{x} \varphi'(H(y))\,dH(y)\right| \ .$$

Assume w.l.o.g. $\varphi' \ge 0$. Otherwise we can treat φ'_+ and φ'_- separately.

$$\int_{x_0}^{x} \varphi'(H(y))\,d\,H(y) = |[\varphi(H(x)) - \varphi(H(x_0))]|$$

$\le K(H(x)(1 - H(x)))^{\delta - \frac{1}{2}}$ since $\varphi(H(x_0))$ is constant.

$$\mathrm{Var}\left|\frac{\beta(x)}{s_N}\right| \le K^2[H(x)(1 - H(x))]^{2\delta-1}\ \mathrm{Var}\ \sqrt{N}(H_N(x) - H(x))$$

$$= K^2\{H(x)(1-H(x))\}^{2\delta-1}\ H(x)(1-H(x))$$

$$\longrightarrow 0 \ \text{as} \ x \to \pm\infty \ .$$

Hence $\frac{\beta(x)}{s_N} \xrightarrow{P} 0$, as $x \longrightarrow \pm\infty$

<u>Proof of (iii)</u> : $D_{iN} = o_p(s_N)$; $i = 1,2,3$.

Consider $$\frac{D_{1N}}{s_N} = -\frac{1}{(N+1)s_N}\int_{-\infty}^{\infty} H_N(x)\varphi'(H(x))\,dC_N(x) \quad .$$

$$(2.2.22) \qquad \left|\frac{D_{1N}}{s_N}\right| \le \frac{1}{Ns_N}\left|\sum_{i=1}^{N}\varphi'(H(X_{Ni}))c_{Ni}\right|$$

$$\le \frac{1}{N}\sum_{i=1}^{N}|V_{Ni}| \qquad \text{where}$$

$$(2.2.23) \qquad V_{Ni} = \frac{c_{Ni}}{s_N}\varphi'(H(X_{Ni})) \quad .$$

Set
$$\begin{aligned} b_i &= N\,, \quad i \le N \\ &= i\,, \quad i > N \\ V_i &= 0\,, \quad i > N\,, \quad V_i = V_{Ni}\,, \quad i \le N\,. \end{aligned}$$

Then (2.2.22) can be written

$$\left|\frac{D_{1N}}{s_N}\right| \le \sum_{i=1}^{\infty}\left|\frac{V_i}{b_i}\right| \quad .$$

We wish to show

$$\frac{1}{b_N}\sum_{i=1}^{N}|V_i| \xrightarrow{P} 0 \quad .$$

This follows from the "particular case 1°," page 241 of Loève [17] if we show

$$\sum_{i=1}^{\infty} \frac{E|V_i|^\alpha}{b_i^\alpha} < \infty \qquad \text{for some } 0<\alpha<1 .$$

Take $\alpha = 2/3$. Then

$$\sum_{i=1}^{\infty} \frac{E|V_i|^{2/3}}{b_i^{2/3}} = \frac{1}{N^{2/3}} \sum_{i=1}^{N} \frac{|C_{Ni}|^{2/3}}{s_N^{2/3}} E|H(X_{Ni})(1-H(X_{Ni}))|^{2/3(\delta-3/2)}$$

$$\le \frac{1}{N^{2/3}} \max_{1\le i\le N} \left|\frac{C_{Ni}}{s_N}\right|^{2/3} \sum_{i=1}^{N} \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{(2/3\delta-1)} dF_{Ni}(x)$$

$$= O(\tfrac{1}{N}) \sum_{i=1}^{N} \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{(2/3\delta-1)} d\, F_{Ni}(x)$$

$$= O(1)\, \frac{1}{N} \sum_{i=1}^{N} \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{(2/3\delta-1)} d\, F_{Ni}(x)$$

$$= O(1) \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{\frac{2\delta}{3}-1} d\, H(x) < +\infty$$

uniformly in N .

Where we have used the fact

$$\max_{1\le i\le N} \left|\frac{C_{Ni}}{s_N}\right|^{2/3} = O(\frac{1}{N^{1/3}}) \quad .$$

Hence $\frac{1}{N} \sum_{i=1}^{N} |V_{Ni}| \xrightarrow{a.s.} 0$

(2.2.24) $\Rightarrow$ $D_{1N} = o_p(s_N)$.

Consider $D_{2N} = \int_{-\infty}^{\infty} (H_N(x)-H(x))\varphi'(H(x))d(C_N(x)-c(x))$.

We note that given any $\epsilon > 0$, $0 < \delta' < \frac{1}{2}$, $\exists C(\epsilon\delta')$ s.t.

$$(2.2.25) \quad P\Big(\sup_{-\infty<x<\infty} \frac{|H_N(x)-H(x)|}{\{H(x)(1-H(x))\}^{\delta-\frac{1}{2}}} > \frac{c(\epsilon,\delta')}{\sqrt{N}}\Big) < \epsilon .$$

Hence on a set of probability $> 1-\epsilon$,

$$|H_N(x) - H(x)||\varphi'(H(x))| \le \frac{KC(\epsilon,\delta')}{\sqrt{N}}|H(x)(1-H(x))|^{\delta-\delta'-1} .$$

Let $\delta^* = \delta - \delta'$, choosing $\delta' < \delta$. Hence, it suffices to show that

$$(2.2.26) \quad \frac{K\,C(\epsilon,\delta')}{s_N\sqrt{N}} \int_{-\infty}^{\infty} |H(x)(1-H(x))|^{\delta^*-1} d(C_N(x)-C(x)) \xrightarrow{P} 0 .$$

We will use the Liapunov criterion for degenerate convergence, (p. 275, B(i) , Loeve [17]) .

$$\frac{K'}{s_N\sqrt{N}} \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{\delta^*-1} dc_N(x) = \frac{K'}{s_N\sqrt{N}} \sum_{i=1}^{N} C_{Ni}\{H(X_{Ni})(1-H(X_{Ni}))\}^{\delta^*}$$

$$(2.2.27) \quad \text{Set } V_{Ni} = \frac{\sqrt{N}\,C_{Ni}}{s_N}\{H(X_{Ni})(1-H(X_{Ni}))\}^{\delta^*-1} .$$

Hence,

$$(2.2.28) \quad \frac{1}{\sqrt{N}\,s_N} \sum_{i=1}^{N} C_{Ni}\{H(X_{Ni})(1-H(X_{Ni}))\}^{\delta^*-1} = \frac{1}{N}\sum_{i=1}^{N} V_{Ni} .$$

(2.2.29) It remains to show $\frac{1}{N} \sum_{i=1}^{N} (V_{Ni} - EV_{Ni}) \xrightarrow{P} 0$. This will be done if we show that for some $\alpha > 0$,

$$(2.2.30) \quad \frac{1}{N^{1+\alpha}} \sum_{i=1}^{N} E|V_{Ni}|^{1+\alpha} \to 0 \quad .$$

Next, choose an $\alpha > 0$ s.t. $(1+\alpha)(\delta^*-1) > -1$ (i.e., $0 < \alpha < \frac{\delta^*}{1-\delta^*}$) . Then,

$$\frac{1}{N^{1+\alpha}} \sum_{i=1}^{N} E|V_{Ni}|^{1+\alpha}$$

$$\leq \frac{1}{N\, N^{\alpha}} \left| \frac{\sqrt{N} \max\limits_{1\leq i\leq N} (C_{Ni})}{s_N} \right|^{1+\alpha} \sum_{i=1}^{N} E\{H(X_{Ni})(1-H(X_{Ni}))\}^{(1+\alpha)(\delta^*-1)}$$

$$= \frac{1}{N^{\alpha}}\, O(1) \frac{1}{N} \sum_{i=1}^{N} \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{(1+\alpha)(\delta^*-1)} d\, F_{Ni}(x)$$

$$= \frac{1}{N^{\alpha}}\, O(1) \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{(1+\alpha)(\delta^*-1)} d\, H(x)$$

$\longrightarrow 0$ as $N \to +\infty$ as the integral is finite.

Thus by C_r-inequality and the Liapunov criterion (2.2.29) follows from (2.2.30).

Again we have used the fact

$$\frac{\sqrt{N} \max\limits_{1\leq i\leq N} |C_{Ni}|}{s_N} = O(1) \quad .$$

Next consider,

$$D_{3N} = \int_{-\infty}^{\infty} \{\varphi(\frac{N}{N+1} H_N(x)) - \varphi(H(x)) - (\frac{N}{N+1} H_N(x) - H(x))$$

$$\times \varphi'(H(x))\} \, d\, C_N(x) .$$

We have to show that

$$\frac{D_{3N}}{s_N} \xrightarrow{P} 0$$

The following substitution will simplify the proof.

(2.2.31) Let $C_{3N} = \frac{D_{3N}}{\sqrt{N}\, s_N}$.

It suffices to prove $C_{3N} = o_p(\frac{1}{\sqrt{N}})$.

Let $Z_{N1} < Z_{N2} < \ldots < Z_{NN}$ be the ordered observations corresponding to $X_{N1}, \ldots, X_{NN}$.

(2.2.32) Let $\lambda_{iN} = |\varphi(\frac{i}{N+1}) - \varphi(H(Z_{Ni}))$

$$- (\frac{i}{N+1} - H(Z_{Ni}))\varphi'(H(Z_{Ni}))| .$$

(2.2.33) Then $|C_{3N}| \le \frac{\max_{1\le i\le N}}{\sqrt{N}\, s_N} \sum_{i=1}^{N} \lambda_{iN} = 0(\frac{1}{N}) \sum_{i=1}^{N} \lambda_{iN}$

where we have used the fact $\frac{\max_{1\le i\le N} |C_{Ni}|}{s_N \sqrt{N}} = 0(\frac{1}{N})$

and the fact that $H_N(Z_{Ni}) = \frac{i}{N}$.

Next consider the function,

$$f(x) = H(x)(1 - H(x)) .$$

Clearly, $f(x) \to 0 \Leftrightarrow x \to \pm \infty$.

Since F_{Ni} are all continuous, given $\epsilon > 0, \exists\, \xi_\epsilon^{(i)} > 0$ s.t.

$$P(f(X_{Ni}) > \frac{\xi_\epsilon^{(i)}}{N}) > 1 - \epsilon\ ,\ i = 1,2,\ldots,N .$$

Let $\xi_\epsilon = \min\limits_{1 \le i \le N} \xi_\epsilon^{(i)}$. Then, $\forall_i = 1,2,\ldots,N$.

$$\text{(2.2.34)} \qquad P(f(X_{Ni}) > \frac{\xi_\epsilon}{N}) > 1 - \epsilon .$$

Let $K_N = [N^{\delta''}]$ where $0 < \delta' < \delta''/2 < \delta/4$. Then (2.2.33) can be written,

$$\text{(2.2.35)} \qquad |C_{3N}| \le C_{3N}^{(1)} + C_{3N}^{(2)} + C_{3N}^{(3)} \quad \text{where}$$

$$\text{(2.2.36)}\ C_{3N}^{(1)} = 0(\tfrac{1}{N}) \sum_{i=1}^{K_N} \lambda_{iN}\ ,\ C_{3N}^{(2)} = 0(\tfrac{1}{N}) \sum_{i=N-K_N+1}^{N} \lambda_{iN}\ ,$$

$$C_{3N}^{(3)} = 0(\tfrac{1}{N}) \sum_{i=K_N+1}^{N-K_N} \lambda_{iN} .$$

(2.2.37) Consider $C_{3N}^{(1)} \le 0(\frac{1}{N}) \sum\limits_{i=1}^{K_N} |\varphi(\frac{i}{N+1})| + 0(\frac{1}{N}) \sum\limits_{i=1}^{K_N} |\varphi(H(Z_{Ni}))|$

$$+\ 0(\tfrac{1}{N}) \sum_{i=1}^{K_N} |\tfrac{i}{N+1} - H(Z_{Ni})|\,|\varphi'(H(Z_{Ni}))| .$$

In (2.2.37), consider the first term;

$$(2.2.38)\quad O(\tfrac{1}{N}) \sum_{i=1}^{K_N} (\varphi \tfrac{i}{N+1}) \mid \le O(\tfrac{1}{N})\ K \sum_{i=1}^{K_N} K \left|\tfrac{i}{N+1}(1-\tfrac{i}{N+1})\right|^{\delta-\frac{1}{2}}$$

$$\le K\ O(\tfrac{1}{N})\ K_N\{\tfrac{N}{(N+1)^2}\}^{\delta-\frac{1}{2}} \le O(\tfrac{1}{N})\ N^{\delta''} \cdot O(\tfrac{1}{N^{\delta-\frac{1}{2}}})$$

$$= O(\tfrac{1}{N^{\frac{1}{2}+\delta-\delta''}}) = o(N^{-\frac{1}{2}}) \quad \text{since} \quad \delta > \delta'' \ .$$

Let $\eta_\epsilon^{(i)} > 0$ be s.t. $P(H(Z_{Ni}) > \frac{\eta_\epsilon^{(i)}}{N}) > 1-\epsilon$.

Let $\eta_\epsilon = \min_{1\le i\le N} \eta_\epsilon^{(i)}$. Then $\forall_i = 1,\ldots,N$.

$$(2.2.38)' \qquad P(H(Z_{Ni}) > \tfrac{\eta_\epsilon}{N}) > 1-\epsilon \ .$$

Hence with probability greater than $1-\epsilon$,

$$O(\tfrac{1}{N}) \sum_{i=1}^{K_N} \varphi(H(Z_{Ni})) < K_N\ O(\tfrac{1}{N})\ K\left|\tfrac{\eta_\epsilon}{N}\ (\tfrac{N-\eta_\epsilon}{N})\right|^{\delta-\frac{1}{2}}$$

$$\le O(\tfrac{1}{N})\quad N^{\delta''} O(N^{-\frac{1}{2}-\delta})$$

$$(2.2.39)\ = O(N^{-\frac{1}{2}}) \quad \text{since} \quad \delta > \delta'' \ .$$

Next note that $\left|\frac{i}{N} - H(Z_{Ni})\right| = |H_N(Z_{Ni}) - H(Z_{Ni})|$.

Next let I_N be the interval in which $H(x)(1-H(x)) > \frac{\xi_\epsilon}{N}$ where ξ_ϵ is given by (2.2.34). Then with prob $> 1-\epsilon$

$$Z_{Ni} \in I_N,\ i = 1,\ldots,N \ ;\ \text{by (2.2.34)} \ .$$

Hence with probability $> 1-\varepsilon$,

$$\left|\frac{i}{N+1} - H(Z_{Ni})\right| |\varphi'(H(Z_{Ni}))|$$

$$\leq K \frac{C(\varepsilon,\delta)}{\sqrt{N}} \left(\frac{\xi_\varepsilon}{N}\right)^{\delta-1} .$$

Hence, with prob $> 1-\varepsilon$

$$O\left(\frac{1}{N}\right) \sum_{i=1}^{K_N} \left|\frac{i}{N+1} - H(Z_{Ni})\right| |\varphi'(H(Z_{Ni}))|$$

$$\leq K \frac{C(\varepsilon,\delta)}{\sqrt{N}} \quad K_N\left(\frac{\xi_\varepsilon}{N}\right)^{\delta-1})\left(\frac{1}{N}\right)$$

$$\leq K \frac{C(\varepsilon,\delta)}{N^{\frac{1}{2}}} \quad N^{\delta''} \left(\frac{\xi_\varepsilon}{N}\right)^{\delta-1} O\left(\frac{1}{N}\right)$$

(2.2.40) $= O(N^{-\frac{1}{2}-\delta+\delta''}) = O(N^{-\frac{1}{2}})$ since $\delta'' < \delta$.

Hence by (2.2.38), (2.2.39) and (2.2.40) we get $C_{3N}^{(1)} = o_p(N^{-\frac{1}{2}})$. To show $C_{3N}^{(2)} = o_p(N^{-\frac{1}{2}})$. Observe that $C_{3N}^{(2)} = O(\frac{1}{N}) \sum_{i=N-K_N+1}^{N} \lambda_{iN}$ and hence has exaxtly the same number of terms (viz K_N) as $C_{3N}^{(1)}$. Hence by symmetry

(2.2.41) $$C_{3N}^{(2)} = o_p(N^{-\frac{1}{2}}) .$$

It remains to prove $C_{3N}^{(3)} = o_p(N^{-\frac{1}{2}})$. Proof of this is exactly as in the Chernoff-Savage theorem, (Puri and Sen [22], pages 402-405).

Set,

$$S_N^{(1)}(\tau) = \{x : \tau \le H(x) \le 1-\tau\} \tag{2.2.42}$$

$$S_N^{(2)}(\tau) = \{x : Z_{N,K_N} < x < H^{-1}(\tau)\} \tag{2.2.43}$$

$$S_N^{(3)}(\tau) = \{x : H^{-1}(1-\tau) < x < Z_{NN-K_N+1}\} \ . \tag{2.2.44}$$

Then,

$$\sqrt{N}\, C_{3N}^{(3)} \le O(1) \sum_{j=1}^{3} \int_{S_N^{(j)}(\tau)} N^{\frac{1}{2}} \Big|\varphi\Big(\frac{N}{N+1} H_N(x)\Big) - \varphi(H(x)) - \Big(\frac{N}{N+1} H_N(x) - H(x)\Big)\varphi'(H(x))\Big| dH_N(x) \ . \tag{2.2.45}$$

Let $v > 0$ be arbitrary but fixed. Then

$$\sup_{|u| \le c} \ \sup_{\tau \le u \le 1-c} \frac{N^{\frac{1}{2}}}{v} \Big|\varphi\Big(u + \frac{v}{\sqrt{N}}\Big) - \varphi(u) - \frac{v}{\sqrt{N}} \varphi'(u)\Big| \to 0 \tag{2.2.46}$$

as $N \to +\infty$ (definition of the derivative and $|\varphi'(u)| \le K|u(1-u)|^{\delta-3/2}$) .

By (2.25), with probability $> 1-\varepsilon$, for any $\varepsilon_1 > 0$, we have,

$$\int_{S_N^{(j)}(\tau)} \sqrt{N} \Big|\varphi\Big(\frac{N}{N+1} H_N(x)\Big) - \varphi(H(x)) - \Big(\frac{N}{N+1} H_N(x) - H(x)\Big)\varphi'(H(x))\Big| d\, H_N(x) < \varepsilon_1 \ . \tag{2.2.47}$$

Using a linear approximation we can write,

$$(2.2.48)\quad \int_{S_N^{(j)}(\tau)} \sqrt{N}\,\Big|\varphi\big(\tfrac{N}{N+1}H_N(x)\big) - \varphi(H(x)) - \big(\tfrac{N}{N+1}H_N(x) - H(x)\big)\varphi'(H(x))\Big|\,dH_N(x)$$

$$\le \int_{S_N^{(j)}(\tau)} \sqrt{N}\,\Big|\tfrac{N}{N+1}H_N(x) - H(x)\Big| \times \Big|\varphi'\big[\alpha H(x) + (1-\alpha)\tfrac{N}{N+1}H_N(x)\big] - \varphi'(H(x))\Big|\,d\,H_N(x)$$

$$j = 2,3,\ 0 < \alpha < 1\,.$$

Because of

$$(2.2.49)\quad \big(\alpha\, H(x) + (1-\alpha)\tfrac{N}{N+1}H_N(x)\big)/H(x) = \alpha + (1-\alpha)\frac{N}{N+1}\,\frac{H_N(x)}{H(x)}$$

and

$$(2.2.50)\quad \frac{1 - \big[\alpha\, H(x) + (1-\alpha)\frac{N}{N+1}H_N(x)\big]}{(1 - H(x))}$$

$$= \alpha + (1-\alpha)\,\frac{\big(1 - \frac{N}{N+1}H_N(x)\big)}{(1 - H(x))}\,,$$

and by (2.2.34) and (2.2.38)$'$, with probability greater than $1 - 2\epsilon$,

$$(2.2.51)\quad \inf_{x \in S_N^{(j)}(\tau)} \frac{\{\alpha H(x) + (1-\alpha)\frac{N}{N+1}H_N(x)\}\{1 - [\alpha H(x) + (1-\alpha)\frac{N}{N+1}H_N(x)]\}}{(H(x)\,(1-H(x)))}$$

$$> \beta_1^2\,\big(\tfrac{N}{N+1}\big)^2\ ;\ j = 2,3\,.$$

$$(2.2.52)\quad \int_{S_N^{(j)}(\tau)} \sqrt{N}\left|\frac{N}{N+1} H_N(x) - H(x)\right| \left|\varphi'[\alpha H(x) + (1-\alpha)\frac{N}{N+1}H_N(x)]\right.$$

$$\left. - \varphi'(H(x))\right| dH_N(x) \le c(\epsilon,\delta')K(1+\beta_{1N}^{-3+2\delta}) \int_{S_N^{(j)}(\cdot)} \{H(x)(1-H(x))\}^{\delta^*-1}$$

$d\,H_N(x)$; $j = 2,3$.

$$\beta_{1N} = \frac{N\beta_1}{N+1} \quad ; \ \delta^* = \delta - \delta'' \ , \ 0 < \delta' < \frac{\delta^*}{2} \ .$$

$$(2.2.53)\quad \sum_{j=2}^{3} \int_{S_N^{(j)}(\tau)} |H(x)(1-H(x))|^{\delta^*-1}\, dH_N(x)$$

$$\le \int_0^\tau |H(x)(1-H(x))|^{\delta^*-1} dH_N(x) + \int_{1-\tau}^{1} (H(x)(1-H(x))|^{\delta^*-1} dH_N(x)$$

$\forall\tau$, $0<\tau<\frac{1}{2}$, we can take expectations of the right hand side.

$$(2.2.54)\quad \int_{(0<x<\tau)\cup(1-\tau<x<1)} H(x)(1-H(x))\}^{\delta^*-1} d\,H(x) \le \frac{2^{1+\delta^*}}{\delta^*}\,\tau^{\delta^*} \ .$$

Given $\epsilon > 0$, δ' , choose τ s.t.

$$\frac{2^{1+\delta^*}}{\delta^*}\, c(\epsilon,\delta')\cdot K(1+\beta_{1N}^{2\delta-3})\,\tau^{\delta^*} < \sqrt{\epsilon} \ .$$

Hence in (2.2.48), the sum for $j = 2,3$ is bounded above in probability by a quantity which can be made arbitrarily small.

By (2.2.47), the same is true for $j = 1$.

Thus we have the result.

Variance Computation:

Clearly, the variance contribution is by $B_{1N} + B_{2N}$

$$B_{1N} = \int_{-\infty}^{\infty} \varphi(H(x))\,d(C_N(x) - C(x))$$

$$B_{2N} = -\int_{-\infty}^{\infty} B^*(x)\,d(H_N(x) - H(x)) \quad \text{where}$$

$$B^*(x) = \int_{x_0}^{x} \varphi'(H(y))\,dc(y) \quad .$$

We can drop $c(x)$ and $H(x)$ since centering does not affect the variance.

(2.2.55) $$\varphi(H(x)) = \int_{x_0}^{x} \varphi'(H(y))\,dH(y) + \varphi(H(x_0)) \quad .$$

(2.2.56) $$\mathrm{Var}(B_{1N} + B_{2N}) = \mathrm{Var}\{\int_{-\infty}^{\infty}\int_{x_0}^{x} \varphi'(H(y))\,d(H(y))\,dC_N(x)$$

$$= \int_{-\infty}^{\infty}\int_{x_0}^{x} \varphi'(H(y))\,dc(y)\,dH_N(x)\} \quad .$$

$$B_{1N} = \sum_{i=1}^{N} \varphi(H(X_{Ni}))c_{Ni}$$

$$B_{2N} = -\frac{1}{N}\sum_{i=1}^{N} B^*(X_{Ni})$$

(2.2.57) $$\text{where } B^*(x) = \int_{x_0}^{x} \varphi'(H(y))\,dc(y)$$

$$= \sum_{j=1}^{N} c_{Ni}\int_{x_0}^{x} \varphi'(H(y))\,dF_{Nj}(y) \quad .$$

(2.2.58) Put $$A_{Ni}(x) = \frac{C_{Ni}}{N} \sum_{j=1}^{N} \int_{x_0}^{x} \varphi'(H(y))\,dF_{Nj}(y) - \frac{1}{N} \sum_{j=1}^{N} C_{Nj} \int_{x_0}^{x} \varphi'(H(y))\,dF_{Nj}(y) .$$

(2.2.59) i.e. $$A_{Ni}(x) = \frac{1}{N} \sum_{j=1}^{N} (C_{Ni}-C_{Nj}) \int_{x_0}^{x} \varphi'(H(y))\,dF_{Nj}(y) .$$

By (2.2.55) , (2.2.56) , (2.2.57) and (2.2.58)

$$B_{1N} + B_{2N} = \sum_{i=1}^{N} A_{Ni}(X_{Ni}) .$$

Hence $\mathrm{Var}(B_{1N} + B_{2N}) = \sum_{i=1}^{N} \mathrm{Var}\, A_{Ni}(X_{Ni})$. The expression (2.2.59) is the same as that given by Hàjek except for the centering.

2.3 Extension of the Main Results:

We have assumed thus far, in both the chapters that the scores a_{Ni} are generated by φ as follows:

$$(2.3.1) \qquad a_{Ni} = \varphi\left(\frac{i}{N+1}\right)$$

However, we are often interested in the limiting behaviour of the simple linear rank statistic S_N when the scores are given by

$$(2.3.2) \qquad a_{Ni} = E\varphi(U_N^{(i)})$$

where $U_N^{(i)}$ is the ith order statistic in a sample of N independent random variables distributed uniformly over $(0,1)$. Hajek [12] has shown that the limiting distribution of S_N is the same whether the scores are given by (2.3.1) or (2.3.2). But as mentioned earlier, Hajek does not give an expression for the centering constant. Hoeffding [14] does find the centering constants. Unfortunately, there is a lack of symmetry in Hoeffding's results in the sense the centering constants are different depending upon whether one uses the scores given by (2.3.1) or (2.3.2), unless one makes the additional assumption.

$$(2.3.3) \qquad |\bar{C}_N| \left(\max_{1 \le i \le N} |C_{Ni} - \bar{C}_N| \right)^{-1}$$

$$\left(\bar{C}_N = \frac{1}{N} \sum_{i=1}^{N} C_{Ni}\right)$$

is bounded. This is rather an unnatural state of affairs because our experience with all the special cases suggests that asymptotically the two statistics should be equivalent in the sense $(S_N - S_N^*)/s_N \to 0$ in probability. We will establish this fact using an approach originally due to Chernoff and Savage. We assume that φ is the inverse of a distribution function. This covers most cases of practical interest and also makes the results much more elegant.

The following construction will simplify our task.

Let the scores be given by,

$$a_{Ni} = a_N(i) = E\varphi(U_N^{(i)}) \quad . \tag{2.3.4}$$

On $0 < t < 1$, we define,

$$\varphi_N(t) = \sum_{i=1}^{[Nt]} a_{Ni} \tag{2.3.5}$$

where $[\alpha]$ = largest integer $\le \alpha$.

For the rest, the notation and terminology are exactly as in the earlier sections of this chapter. The following lemma is helpful.

<u>Lemma 2.3.1</u>: <u>Let φ be the inverse of a distribution function and also satisfy the conditions of theorem 2.2.1. In addition, let</u>

$$(2.3.5) \qquad \max_{1\le i\le N} \frac{|c_{Ni}|}{s_N} = O(\frac{1}{\sqrt{N}}) .$$

Then

(i) $\lim_{N\to+\infty} \varphi_N(t) = \varphi(t)$

(ii) $| \int_{-\infty}^{\infty} \{\varphi_N(\frac{N}{N+1} H_N(x)) - \varphi(\frac{N}{N+1} H_N(x))\} dC_N(x) = o_p(s_N)$

Proof: We will prove (ii) as the proof of (i) can be found in Puri and Sen [22], pages 408-409.

Proof of (ii): Recall that

$$|C_N(x)| = |\sum_{i=1}^{N} c_{Ni}\, [I(X_{Ni} \le x)| \le \max_{1\le i\le N} |c_{Ni}| \sum_{i=1}^{N} I(X_{Ni} \le x)$$

Hence,

$$(2.3.7) \qquad |\int_{-\infty}^{\infty} \{\varphi_N(\frac{NH_N(x)}{N+1}) - \varphi(\frac{NH_N(x)}{N+1})\} dC_N(x)|$$

$$\le \max_{1\le i\le N} |c_{Ni}| \sum_{i=1}^{N} |\varphi_N(\frac{N}{N+1} H_N(X_{Ni})) - \varphi(\frac{N}{N+1} H_N(X_{Ni}))|$$

$$= \max_{1\le i\le N} |c_{Ni}| \sum_{i=1}^{N} |\varphi_N(\frac{i}{N+1}) - \varphi(\frac{i}{N+1})| .$$

Now,

$$(2.3.8) \qquad \max_{1\le i\le N} \frac{|c_{Ni}|}{s_N} \sum_{i=1}^{N} |\varphi_N(\frac{i}{N+1}) - \varphi(\frac{i}{N+1})|$$

$$= O(\frac{1}{\sqrt{N}}) \sum_{i=1}^{N} |\varphi_N(\frac{i}{N+1}) - \varphi(\frac{i}{N+1})|$$

$$= O(1)\; N^{-\frac{1}{2}} \sum_{i=1}^{N} |\varphi_N(\frac{i}{N+1}) - \varphi(\frac{i}{N+1})|$$

Set

$$(2.3.9) \qquad A_N = N^{-\frac{1}{2}} \sum_{i=1}^{N} \left|\varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right)\right| .$$

In view of (2.3.7) and (2.3.8) (ii) will be established if we can show that

$$\lim_{N \to +\infty} A_N = 0 .$$

This is done in Puri and Sen [16], pages 409-411, and the lemma is proved.

Consider, with the scores given by (2.3.2)

$$(2.3.10) \qquad S_N^* = \sum_{i=1}^{N} C_{Ni} a_N(R_{Ni}) .$$

Let S_N be defined as before. Clearly,

$$(2.3.11) \qquad S_N^* = \sum_{i=1}^{N} C_{Ni} \varphi_N\left(\frac{R_{Ni}}{N+1}\right)$$

$$= \int_{-\infty}^{\infty} \varphi_N\left(\frac{N}{N+1} H_N(x)\right) dC_N(x)$$

<u>Theorem 2.3.2</u>: <u>With the definitions given, let</u> φ <u>satisfy conditions of lemma</u> 2.3.1. <u>Also let</u> $\max_{1 \le i \le N} |C_{Ni}| s_N^{-1} = O\left(\frac{1}{\sqrt{N}}\right)$. <u>Then,</u> $\mathcal{L}\left(\frac{S_N^* - \mu_N}{s_N}\right) \to \mathcal{N}(0,1)$, <u>where</u> μ_N <u>is the same as that for</u> S_N .

<u>Proof:</u> We can write,

$$S_N^* = S_N + (S_N^* - S_N)$$

Since we have already established in theorem 2.2.1 the fact that

$$\mathcal{L}\left(\frac{S_N - \mu_N}{s_N}\right) \to \mathcal{N}(0,1)$$

it suffices to prove that

(2.3.12) $$\frac{S_N - S_N^*}{s_N} \to 0 \quad \text{in probability.}$$

Then, Polya's theorem gives

$$\mathcal{L}\left(\frac{S_N^* - \mu_N}{s_N}\right) \to \mathcal{N}(0,1)$$

But by (2.3.11), we have

$$\left|\frac{S_N - S_N^*}{s_N}\right| = \frac{1}{s_N}\left|\int_{-\infty}^{\infty}\{\varphi_N(\frac{N}{N+1} H_N(x)) - \varphi|\frac{N}{N+1} H_N(x) dC_N(x)\right|$$

$= o_p(1)$ by Lemma 2.3.1. This proves (2.3.12) and hence the theorem is proved.

CHAPTER III

Multivariate Extension:

3.1 Preliminaries: In this chapter we extend the results of chapter I for the case when the observations $X_{N1}, \ldots, X_{NN}$ are p-variate random vectors. We use the Cramér - Wold technique for this purpose. A multivariate extension along the lines of Hajek's original proof as well as some testing procedures based on it can be found in Puri and Sen [21].

3.2 Notation and Terminology: Let $\underset{\sim}{X}_{N1}, \ldots, \underset{\sim}{X}_{NN}$ be a sequence independent p-variate random vectors with continuous distribution functions $F_{N1}, \ldots, F_{NN}$. To be more explicit,

$$\underset{\sim}{X}_{Ni} = (X_{Ni}^{(1)}, \ldots, X_{Ni}^{(p)})' \quad ; \quad 1 \leq i \leq N . \tag{3.2.1}$$

Let
$$\underset{\sim}{x} = (x^{(1)}, \ldots, x^{(p)})' .$$

Then

$$F_{Ni}(\underset{\sim}{x}) = P(X_{Ni}^{(1)} \leq x^{(1)}, \ldots, X_{Ni}^{(p)} \leq x^{(p)}) \; ; \; 1 \leq i \leq N . \tag{3.2.2}$$

Consider next the random matrix $\underset{\sim}{X}_N$ corresponding the

sample of random vectors $\underset{\sim}{X}_{N1}, \ldots, \underset{\sim}{X}_{NN}$ defined by

$$\underset{\sim}{X}_N = \begin{bmatrix} X_{N1}^{(1)} & \ldots & X_{Ni}^{(1)} & \ldots & X_{NN}^{(1)} \\ X_{N1}^{(2)} & \ldots & X_{Ni}^{(2)} & \ldots & X_{NN}^{(2)} \\ X_{N1}^{(\nu)} & \ldots & X_{Ni}^{(\nu)} & \ldots & X_{NN}^{(\nu)} \\ X_{N1}^{(p)} & \ldots & X_{Ni}^{(p)} & \ldots & X_{NN}^{(p)} \end{bmatrix} \tag{3.2.3}$$

In the matrix $\underset{\sim}{X}_N$ observe that each row $(X_{N1}^{(\nu)}, X_{N2}^{(\nu)}, \ldots, X_{NN}^{(\nu)})$, $1 \le \nu \le p$ is composed of independent random variables.

Let the marginal distribution function of the entry $X_{Ni}^{(\nu)}$ be $F_{Ni}^{(\nu)}$; $1 \le \nu \le p$, $1 \le i \le N$.

i.e., for any x, $-\infty < x < +\infty$.

$$F_{Ni}^{(\nu)}(x) = P(X_{Ni}^{(\nu)} \le x) \tag{3.2.4}$$

We now proceed to define the ranks for each row.

Consider the νth row $(X_{N1}^{(\nu)}, \ldots, X_{NN}^{(\nu)})$. For the random variable $X_{Ni}^{(\nu)}$, we define its rank

$$R_{Ni}^{(\nu)} = (\text{Number of } X_{Nj}^{(\nu)} \le X_{Ni}^{(\nu)}), \quad 1 \le \nu \le p. \tag{3.2.5}$$

Thus corresponding to the matrix $\underset{\sim}{X}_N$, we have the rank matrix $\underset{\sim}{R}_N$ defined by

$$\underset{\sim}{R}_N = \begin{bmatrix} R_{N1}^{(1)} & R_{N2}^{(1)} & \cdots & R_{Nj}^{(1)} & \cdots & R_{NN}^{(1)} \\ R_{N1}^{(2)} & R_{N2}^{(2)} & \cdots & R_{Nj}^{(2)} & \cdots & R_{NN}^{(2)} \\ R_{N1}^{(\nu)} & R_{N2}^{(\nu)} & \cdots & R_{Nj}^{(\nu)} & \cdots & R_{NN}^{(\nu)} \\ \vdots & \vdots & & \vdots & & \vdots \\ R_{N1}^{(p)} & R_{N2}^{(p)} & \cdots & R_{Nj}^{(p)} & \cdots & R_{NN}^{(p)} \end{bmatrix} \tag{3.2.6}$$

Let $\{C_{N1}^{(\nu)}, \ldots C_{NN}^{(\nu)}\}$, $1 \le \nu \le p$ be p-sets of known (regression) constants.

Let $\{A_{N1}^{(\nu)}, \ldots A_{NN}^{(\nu)}\}$, $1 \le \nu \le p$ be p-sets of known constants (scores).

We define the simple linear rank vector $\underset{\sim}{S}_N$ corresponding to $\underset{\sim}{X}_N$ by

$$\underset{\sim}{S}_N = (S_N^{(1)}, \ldots, S_N^{(p)})' \text{ where} \tag{3.2.7}$$

$$S_N^{(\nu)} = \sum_{i=1}^{N} C_{Ni}^{(\nu)} A_N^{(\nu)} (R_{Ni}^{(\nu)}) \tag{3.2.8}$$

We are interested in the Asymptotic normality of the simple linear rank vector $\underset{\sim}{S}_N = (S_N^{(1)}, \ldots, S_N^{(p)})'$.

As before we assume that the scores $A_{Ni}^{(\nu)}$ are generated by known functions φ_ν in either of the following ways:

(3.2.9) $$A_{Ni}^{(\nu)} = \varphi_\nu\left(\frac{i}{N+1}\right)$$

(3.2.10) $$A_{Ni}^{(\nu)} = E\,\varphi_\nu(U_N^{(i)})$$

with the usual notation.

3.3 <u>Main</u> <u>Results</u>: In this section we establish the asymptotic multinormality of the statistic $\underset{\sim}{S}_N$ defined by (2.7) and (2.8). We make the following assumptions:

For each ν, $1 \le \nu \le p$, there exist constants $\delta_\nu > 0$ and generic constants K such that

(i) $|\varphi_\nu(t)| \le K\{t(1-t)\}^{\delta_\nu - 1/2}$, $0 < t < 1$

(ii) $|\varphi_\nu'(t)| \le K\{t(1-5)\}^{\delta_\nu - 3/2}$, $0 < t < 1$

(iii) For each ν, $1 \le \nu \le p$, the empirical process $X_N^{(\nu)}(t)$ and ahe weighted empirical process $Y_N^{(\nu)}(t)$ defined as in chapter I, with covariance kernels $K_N^{(\nu)}(s,t)$ and $R_N^{(\nu)}(s,t)$ respectively, satisfy conditions of theorem (1.3.2) (or corollary (1.3.3)).

We then have the following theorem:

Theorem 3.3.1: *Let conditions* (i), (ii) *and* (iii) *above be satisfied. Let, in addition*

(3.3.1)
$$\frac{\max\limits_{1 \le i \le N} |c_{Ni}^{(\nu)}|}{s_\nu} = O\left(\frac{1}{\sqrt{N}}\right) \; ; \; 1 \le \nu \le p$$

(3.3.2)
$$s_\nu^2 = \text{Var } S_N^{(\nu)} \; ; \; 1 \le \nu \le p \; .$$

We have dropped the index N in order not to encumber our notation too much. Otherwise, note that s_ν depends on N.

Then, for every vector $\underset{\sim}{\lambda}$ in R^p, $\lambda'(\underset{\sim}{S}_N - \underset{\sim}{\mu}_N)/(\underset{\sim}{\lambda}'\Sigma_N \underset{\sim}{\lambda})^{1/2}$ has *asymptotically a standard normal distribution where* $\underset{\sim}{\mu}_N$ *and* Σ_N *are defined below:*

(3.3.3)
$$\underset{\sim}{\mu}_N = (\mu_N^{(1)}, \ldots, \mu_N^{(p)})'$$

(3.3.4)
$$\mu_N^{(\nu)} = \sum_{i=1}^{N} c_{Ni}^{(\nu)} \int_{-\infty}^{\infty} \varphi\left(H^{(\nu)}(x)\right) dF_{Ni}^{(\nu)}(x)$$

(3.3.5)
$$H^{(\nu)}(x) = \frac{1}{N} \sum_{i=1}^{N} F_{Ni}^{(\nu)}(x)$$

(3.3.6) $$\Sigma_N = (s_{\mu\nu}) \; ; \; 1 \le \mu,\nu \le p$$

(3.3.7) $$s_{\mu\nu} = \mathrm{Cov}\,(S_N^{(\mu)}, S_N^{(\nu)})$$

As in the univariate case, the variance-covariance Matrix Σ_N can be taken to consist of approximate variance-covariance terms, namely,

(3.3.8) $$s_{\mu\nu} = \sum_{i,k=1}^{N} \mathrm{Cov}\left[A_{Ni}^{(\mu)}(X_{Ni}^{(\mu)}),\, A_{Nk}^{(\nu)}(X_{Nk}^{(\nu)})\right]$$

<u>where</u>

(3.3.9) $$A_{Ni}^{(\nu)}(x) = \frac{1}{N}\sum_{j=1}^{N} (C_{Nj}^{(\nu)} - C_{Ni}^{(\nu)}) \int_{-\infty}^{\infty} \{I(x \le y) - F_{Ni}^{(\nu)}(y)\}$$

$$\times\, \varphi_\nu'\left(H^{(\nu)}(y)\right) dF_{Nj}^{(\nu)}(y)$$

$$1 \le \mu,\nu \le p$$

<u>Proof</u>: We use the Crámer-Wold criterion: Let A_1, $A_2, \ldots, A_p$ be fixed but arbitrary constants. It suffices to show that,

(3.3.10) $$A_1 S_N^{(1)} + A_2 S_N^{(2)} + \ldots + A_p S_N^{(p)} = U_N \,,$$

say, is asymptotically normal.

Observe that by theorem 1.6.1 of chapter I, $A_\nu S_N^{(\nu)}$, properly normalized is asymptotically normal. In fact, we have shown slightly more. To be more precise, we have established that

$$\frac{S_N^{(\nu)} - \mu_N^{(\nu)}}{s^{(\nu)}} = (X_N^{(\nu)} - Y_N^{(\nu)}) + \frac{D_N^{(\nu)}}{s^{(\nu)}}$$

where $$D_N^{(\nu)} = \sum_{i=1}^{3} D_{iN}^{(\nu)}$$

$$(X_N^{(\nu)} - Y_N^{(\nu)}) \xrightarrow{\mathcal{L}_2} (X^{(\nu)} - Y^{(\nu)}) \text{ and}$$

$$\frac{D_N^{(\nu)}}{s_\nu} \xrightarrow{P} 0$$

Thus,

$$\frac{S_N^{(\nu)} - \mu_N^{(\nu)}}{s_\nu} \xrightarrow{P} X^{(\nu)} - Y^{(\nu)} \quad ; 1 \le \nu \le p$$

which is Gaussian with mean 0 and variance 1 . Hence,

$$\sum_{\nu=1}^{p} A_\nu \left(\frac{S_N^{(\nu)} - \mu^{(\nu)}}{s_\nu}\right) \xrightarrow{P} \sum_{\nu=1}^{p} A_\nu (X^{(\nu)} - Y^{(\nu)})$$

which is Gaussian. This completes the proof.

Computation of Σ_N : As before, we can disregard the $\frac{D_N^{(\nu)}}{s_\nu}$ terms and express each $S_N^{(\nu)}$ as the sum of independent random variables.

(3.3.11) $$S_N^{(\nu)} - \mu_N^{(\nu)} - D_N^{(\nu)} = \sum_{i=1}^{N} A_{Ni}^{(\nu)} \left(X_{Ni}^{(\nu)}\right) \quad \text{where}$$

$$A_{Ni}^{(\nu)}(x) = \frac{1}{N} \sum_{j=1}^{N} (C_{Ni} - C_{Nj}) \int_{-\infty}^{\infty} I(X_{Ni}^{(\nu)} \leq y)\, \varphi^{1}\left(H^{(\nu)}(y)\right) dF_{Nj}^{(\nu)}(y)$$

Similarly,

(3.3.12) $$S_N^{(\mu)} - \mu_N^{(\mu)} - D_N^{(\mu)} = \sum_{k=1}^{N} A_{Nk}^{(\mu)} \left(X_{Nk}^{(\mu)}\right)$$

In view of the fact $D_N^{(\nu)} = o_p(s_\nu)$, we have

(3.3.13) $$\mathrm{Cov}\,(S_N^{(\mu)}, S_N^{(\nu)}) \doteq s_{\mu\nu}$$

$$= \sum_{i,k=1}^{N} \mathrm{Cov}\left[A_{Nk}^{(\mu)} \left(X_{Nk}^{(\mu)}\right),\ A_{Ni}^{(\nu)} \left(X_{Ni}^{(\nu)}\right)\right]$$

where all the quantities have already been defined.

Chapter IV

DECOMPOSITIONS OF GAUSSIAN NOISE :

4.1 Introduction: This chapter will be devoted to an examination of the phenomenon of "Gaussian Noise". Specifically, the "noises" giving rise to two well-known Gaussian stochastic processes, viz, the Wiener process (or Brownian Motion) and the Borwnian Bridge process (or the Doob-Kac process) will be studied. Such noises will be obtained as the limits in a certain sense of "discrete" noises. A discussion of the phenomenon of noise, particularly of white noise can be found in most advanced text books on stochastic processes among which Arnold [2] , Gel'fand and Vilenkin [8] and Hida [13] merit special mention. Hida in particular gives an extensive treatment of white noise unlike most other books, which treat it in a cursory manner. Our discussion will differ somewhat from all these in the sense we view Gaussian Noise as the limiting action of a simpler phenomenon. Rest of this section will be devoted to a brief description of noise, using white noise as example.

White noise is understood in the applied sciences to be a stationary Gaussian process $\{\xi(t) : 0 \leq t < +\infty\}$ with covariance kernel

$$C(t) = E[\xi(s)\xi(s+t)] , \quad E\,\xi(t) = 0 \tag{4.1.1}$$

and having a constant "spectral intensity"

$$(4.1.2) \quad f(\lambda) = \frac{1}{2\pi} \int_0^\infty C(t)\, e^{-i\lambda t} dt = \frac{a}{2\pi}$$

where 'a' is a constant. Thus in the spectrum of such a process, all the frequencies participate at the same intensity just as in the case of the spectrum of white light. Hence the name white noise. But the identity (4.1.2) is compatible only when $C(t)$ is the so called Dirac "delta function" and such a process does not exist in the usual sense of a stochastic process. But $\xi(t)$ does arise in physical problems in the following manner:

Let $X(t)$ be the state of a physical system at time t. It often arises as the solution of a differential equation of the type

$$(4.1.3) \quad d\,X(t) = f(t, X(t)dt + G(t, X(t)dW(t)$$
$$= f(t, X(t))dt + G(t, X(t))\xi(t)dt$$

where $W(t)$ is the Wiener process. Recall that the Wiener process $W(t)$ is a Gaussian process with $E\,W(t) \equiv 0$ and $K(s,t) = E(X(s)X(t)) = \min(s,t)$. In this sense white noise is often interpreted as the "derivative" of the Wiener process. However, since the sample paths of the Wiener process are nowhere differentiable with probability 1, we are again left with the conclusion that $\xi(t)$ does not exist as

a stochastic process. It should be noted that the choice of the Wiener process (and of white noise) in equations of the type (4.13) is largely one of mathematical expediency and often physically unrealistic. The definition of white noise can be made rigorous in the following manner:

Let S denote the space of all infinitely differentiable function $\varphi(t)$, $t \geq 0$ with compact support. That is, for each $\varphi \in S$, there is a compact set $K_{N,\varphi}$, $n = 0,1,\ldots$ such that $t \notin K_{n,\varphi}$ entails $\varphi^{(n)}(t) = 0$, $n = 0,1,\ldots$. We topologize S in the following manner. A sequence $\{\varphi_n\}$ in S is said to converge to a function φ in S if $\{\varphi , \varphi_n\}$ vanish outside the same compact set K and

$$(4.14) \qquad \lim_{n\to+\infty} \{ \max_t |\varphi_n^{(k)}(t) - \varphi_n^{(k)}(t)\} = 0 \ ; \quad k = 0,1,2,\ldots \ .$$

<u>Definition</u> : Let $\Lambda : S \to \mathbb{R}^1$ be a continuous linear functional. Then Λ is called a "distribution" or a "generalized function" in S .

<u>Definition</u> : Let Λ be a generalized function on S . Then its "distributional derivative" (generalized derivative) is the map

$$(4.1.5) \qquad \Lambda'\varphi = -\Lambda\varphi' \quad .$$

<u>Definition</u> : A "generalized stochastic process" is a stochastic process $\{\Phi(\varphi) : \varphi \in S\}$ in the following sense:

(i) $\forall\varphi \in S$, $\Phi(\varphi)$ is a random variable

(ii) If φ_1 , $\varphi_2 \in S$ and α_1, α_2 arbitrary constants,

$$\Phi(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1\Phi(\varphi_1) + \alpha_2\Phi(\varphi_2)$$

with probability 1 .

(iii) If $\varphi_{kn} \to \varphi_k$, $k = 1,\ldots,m$ in S , then the random vector

$$(\Phi(\varphi_{1n}),\ldots,\Phi(\varphi_{mn})) \to (\Phi(\varphi_1),\ldots,\Phi(\varphi_m))$$

in distribution.

We shall now define a simple transformation which enables us to obtain $W(t)$ and $\xi(t)$ as generalized stochastic processes. We first need the following definitions.

<u>Definition</u> : A real-valued function f on $[0,\infty)$ is said to be "locally integrable" if

(4.1.6) $$\int_K |f(x)| d\,x < +\infty$$

for every compact set K .

Observe that every locally integrable function f defines a generalized function Λf on S whose "action" is given by , $\langle \Lambda f, (\cdot) \rangle$ where

(4.1.7) $$\langle \Lambda f , \varphi \rangle = (\Lambda f)(\varphi) = \int_0^\infty f(t)\varphi(t)dt \, .$$

Henceforth we shall find it convenient to use the symbol $\langle \Lambda, (\cdot) \rangle$ to mean $\Lambda(\cdot)$.

<u>Definition</u> : The "derivative" of a locally integrable function f is defined to the distributional derivative Λf . That is to say ,

$$(4.1.8) \qquad \langle \Lambda' f \, , \, \varphi \rangle = -\langle \Lambda f, \varphi' \rangle = -\int_0^\infty \varphi'(t) f(t) dt \; .$$

Let $W(t)$ be the Wiener process. Then the "Generalized Wiener Process" $\{W(\varphi) \, : \, \varphi \in S\}$ is obtained by the transformation,

$$(4.1.9) \quad \langle \Lambda W \, , \, \varphi \rangle = W(\varphi) = \int_0^\infty \varphi(t) W(t) dt \; .$$

<u>Definition</u> : The white noise ξ is defined to be the distributional derivative of the generalized Wiener process defined by (4.19). That is to say, its action $\Lambda\xi$ is given by ,

$$(4.1.10) \qquad \langle \Lambda\xi \, , \, \varphi \rangle = \langle \Lambda' W, \varphi \rangle = \langle \Lambda W, \varphi' \rangle$$

$$= - \int_0^\infty W(t) \varphi'(t) dt \quad .$$

Observe that (4.1.10) does not permit us to view white noise as a random function since Λf is <u>not</u> a function but a generalized function. But $\Lambda\xi$ does exist as a generalized stochastic process. This fact as well as many of the

properties of $\Lambda\xi$ can be found in Gel'fand and Vilenkin [8] . Hence we will not go into these matters but restrict ourselves to the following remarks.

The transformations Λ and Λ' have in effect resulted in "smoothing" W and ξ . Thus we are in a position to discuss their differentiability properties. But this has been obtained at a certain preice as we have lost sight of the time parameter t as well as the sample path descriptions of W(t) . One often views a stochastic process as the state of a system as it evolves in time. Many of the most important properties of stochastic properties, such as for example the Markov property make explicit use of this concept. Thus a descritpion of the sample paths is well nigh indispensible for any analysis of stochastic processes. This undoubtedly accounts for the fact that inspite of their desirable analytical properties, generalized random functions have not found extensive use in the study of stochastic differential equations.

In the next section we will introduce a family of random transformations of the type (4.1.7) in an attempt to incorporate some of the advantages of generalized random functions without losing the sample path characteristics. A consequence of this will be a central limit theorem for white noise analogous to the Donsker theorem for random walks. Then we will obtain a different but equivalent description of white noise. In section 3, we apply the same methods to the noise giving rise to the Brownian Bridge

process. This in turn yields a rather remarkable connection between the projection techniques of Chernoff and Savage (used by us in Chapter II) and a canonical approximation for Gaussian noise.

4.2 Approximations of White Noise :

Henceforth we shall restrict ourselves to the set $T = [0,1]$ which is compact. Then the space S becomes the space of all infinitely differentiable functions on T. At the endpoints of T, we take one-sided derivatives.

Let $\mathfrak{F}$ be the class of all bounded measurable functions on T. Observe that since T is compact, every f in $\mathfrak{F}$ is Lebesgue integrable.

For each f in $\mathfrak{F}$, we define a map Λf on $S \times T$ given by

$$(\varphi, t) \to \int_0^t f(s)\varphi(s)\,ds \,. \tag{4.2.1}$$

We shall use the sumbols $\langle \Lambda_t f, \varphi \rangle$ and $\langle \Lambda f, \varphi \rangle(t)$ equivalently to mean,

$$\langle \Lambda_t f, \varphi \rangle = \langle \Lambda f, \varphi \rangle(t) = \int_0^t \varphi(s) f(s)\,ds \,. \tag{4.2.2}$$

With the operator Λ we associate its "differential" operator $\hat{\Lambda}$ whose action is given by ,

$$(4.2.3) \qquad \langle \hat{\Lambda}_t f \, , \, \varphi \rangle = \langle \hat{\Lambda} f \, , \, \varphi \rangle (t)$$

$$= \varphi(t) f(t) - \int_0^t f(s) \varphi'(s) ds \; .$$

Since $[0,t] \subset [0,1]$ is compact and since f is bounded, the linearity and continuity of $\Lambda_t f$ are immediate. Thus $\Lambda_t f$ is a generalized function on S for each t and f.

<u>Remark</u> : If we take $T = [0,\infty)$ as before then require φ to have compact support, the definition (4.23) coincides with the definition of the distributional derivative of Λf given in the preceding section. (Because, f is bounded, support of φ is compact and hence,

$$\lim_{t \to +\infty} \{\varphi(t) f(t) - \int_0^t f(s) \varphi'(s) ds\} = -\int_0^\infty f(s) \varphi'(s) ds$$

$$= \langle \Lambda' f, \varphi \rangle) \; .$$

Let $\{X(t) \; : \; 0 \le t \le 1\}$ be a stochastic process which is integrable in t. Then

$$(4.2.4) \qquad \langle \Lambda_t X, \varphi \rangle = \langle \Lambda X, \varphi \rangle (t) = \int_0^t X(s) \varphi(s) ds$$

is also a stochastic process.

Let the sample paths of $X(t)$ be bounded measurable functions. This includes the space $C[0,1]$ and $D[0,1]$ and thus covers most cases of interest. Then the "derivative"

of $X(t)$ is defined by

$$(4.2.5)\qquad \langle \hat{\Lambda}_t X, \varphi\rangle = \langle \hat{\Lambda} X , \varphi\rangle(t) = \varphi(t)X(t) - \int_0^t \varphi'(s)X(s)ds$$

and is also a stochastic process. Thus our definition has the advantage that with reasonable sample path restrictions, stochastic processes are closed under differentiation.

Let $\{W(t) : 0 \le t \le 1\}$ be the Wiener process. Then

$$(4.2.6)\qquad \langle \Lambda W, \varphi\rangle(t) = \int_0^t W(s)\varphi(s)ds \quad .$$

We define white noise by the operator $\Lambda\xi = \hat{\Lambda} W$ whose "action" is given by

$$(4.2.7)\qquad \langle \Lambda\xi, \varphi\rangle(t) = \langle \hat{\Lambda} W, \varphi\rangle(t)$$

$$= \varphi(t)W(t) - \int_0^t \varphi'(s)W(s)ds \quad .$$

Again if we let $t \to +\infty$, with the usual restrictions on φ , the definition (4.2.7) coincides with that given for white noise in (4.1.10) in the preceding section.

Physically we can interpret (4.2.7) as the "resultant" state of a system (or a measuring device) whose character (such as friction) is given by the function φ , when acted upon for a period t by an external source of white noise.

Observe that the families $\{\Lambda_t W ; 0 \le t \le 1\}$ and

$\{\Lambda_t \xi : 0 \le t \le 1\}$ are not stochastic processes but act on the function space S to generate families of stochastic processes $\langle \Lambda_t W, \varphi \rangle$ and $\langle \Lambda_t \xi, \varphi \rangle$, $\varphi \in S$. For each particular φ, we have a stochastic process; for each fixed t, we have a generalized stochastic process over S.

Next we turn our attention to "discrete" processes.

Let $Y_1, Y_2, \ldots, Y_N, \ldots$, be independent, identically distributed random variables. Without loss of generality we assume

$$E\, Y_i = 0\ , \quad \mathrm{Var}\, Y_i = 1\ .$$

We define the discrete process $\{X_n(t)\ ;\ 0 \le t \le 1\}$ corresponding to the sequence $\{Y_k\}$ by

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} Y_i \tag{4.2.8}$$

where $[\alpha]$ is the greatest integer $\le \alpha$.

It is well known that $X_n(t)$ is realizable in the function space $D[0,1]$ (or simply D) defined as

<u>Definition</u>: D is the class of all functions $x(t)$, $0 \le t \le 1$, such that $x(t^-)$ and $x(t^+)$ exist for all $0 < t < 1$ and $x(t) = x(t^+)$. Further,

$$x(0) = x(0+) \quad \text{and} \quad x(1) = x(1-)\ . \tag{4.2.9}$$

For any two functions x and y in D, we define the metric ρ where

$$(4.2.10) \qquad \rho(x,y) = \max_{0\le t\le 1} |x(t) - y(t)| \ .$$

<u>Definition</u> : We now define the operator $\Lambda_t X_n$ by its action S, viz,

$$(4.2.11) \qquad \langle \Lambda_t X_n, \varphi\rangle = \langle \Lambda X_n, \varphi\rangle(t) = \int_0^t \varphi(s) X_n(s)\,ds \ .$$

<u>Definition</u> : We define the "discrete noise" ξ_n corresponding to the process $X_n(t)$ (or the sequence $\{Y_k\}$) by its action,

$$(4.2.12) \qquad \langle \Lambda_t \xi_n, \varphi\rangle = \langle \Lambda \xi_n, \varphi\rangle(t)$$

$$= \langle \hat{\Lambda} X_n, \varphi\rangle(t) = \varphi(t) X_n(t) - \int_0^t \varphi'(s) X_n(s)\,ds \ .$$

<u>Remark</u> : Observe that the processes $\langle \Lambda_t X_n, \varphi\rangle$ and $\langle \Lambda_t \xi_n, \varphi\rangle$ are also realized in $D[0,1]$. Again we note that $\Lambda_t X_n$ and $\hat{\Lambda}_t X_n = \Lambda_t \xi_n$ are not processes in the conventional sense.

We next examine the limiting behaviour of ξ_n .

<u>Theorem 4.2.1</u> : <u>Let</u> $\{Y_k\}$ <u>be independent, identically distributed random variables with mean</u> 0 <u>and variance</u> 1 .

<u>Let</u> $X_n(t)$ <u>and</u> ξ_n <u>be as defined. Then there exists a sequence</u> $\{\tilde{\xi}_n\}$ <u>of discrete noises on the same probability space whose action</u> $\langle \Lambda_t \tilde{\xi}_n, \varphi \rangle$ <u>has the same distribution as the action</u> $\langle \Lambda_t \xi_n, \varphi \rangle$ of ξ_n <u>and converge weakly to the action of a white noise on the same probability space</u>.

<u>Proof</u> : For weak convergence of stochastic processes we need to prove two things: convergence of the finite dimensional distributions and the convergence of sample paths. We shall follow section 13.4 of Breiman [4] .

First we shall establish the continuity of $\langle \hat{\Lambda}_t \cdot, \varphi \rangle$ on $D[0,1]$ for $(t,\varphi) \in [0,1] \times S$.

Recall that, $\forall x \in D$,

$$\langle \hat{\Lambda}_t x, \varphi \rangle = x(t)\varphi(t) - \int_0^t x(s)\varphi'(s)\,ds \; .$$

Let x and y be elements in D . Then

$$(4.2.13) \qquad |\langle \hat{\Lambda}_t x, \varphi \rangle - \langle \hat{\Lambda}_t y, \varphi \rangle|$$

$$= |\varphi(t)(x(t)-y(t)) - \int_0^t (x(s)-y(s))\varphi'(s)\,ds|$$

$$\leq |x(t) - y(t)||\varphi(t)| + \int_0^t |x(s) - y(s)||\varphi'(s))\,ds \; .$$

Let $M = \max\{\max_{0\leq t\leq 1} |\varphi(t)|, \max_{0\leq t\leq 1} |\varphi'(t)\}$. Then in (4.2.13)

$$|\langle\hat{\Lambda}_t x,\varphi\rangle - \langle\hat{\Lambda}_t y,\varphi\rangle| \le M(\max_{0\le s\le t}|x(s)-y(s)| + 1 \max_{0\le s\le t}|x(s)-y(s)|)$$

$$\le M 2\rho(x,y) .$$

Thus, given $\varepsilon > 0$, choose $\delta < \frac{\varepsilon}{2M}$. Then, $\rho(x,y) < \delta$

$$|\langle\hat{\Lambda}_t x,\varphi\rangle - \langle\hat{\Lambda}_t y,\varphi\rangle| < \varepsilon . \tag{4.2.14}$$

Thus $\langle\hat{\Lambda}_t \cdot, \varphi\rangle$ is continuous on $D[0,1]$.

By theorem 1.3.8 Breiman [4], there exists a process $\tilde{X}_n(t)$ having the same distribution as $X_n(t)$ and a Wiener process $W(t)$ all on the same probability space such that for any subsequence $\{n_k\}$ increasing rapidly enough,

$$\sup_{0\le t\le 1}|\tilde{X}_{nk}(t) - W(t)| \to 0 \quad \text{a.s.}$$

Let $\tilde{\xi}_n$ be the noise corresponding to $\tilde{X}_n(t)$. Since $X_n(t)$ and $\tilde{X}_n(t)$ have the same distribution, it is obvious that $\langle\hat{\Lambda}_t X_n,\varphi\rangle = \langle\Lambda_t \xi_n,\varphi\rangle$ and $\langle\hat{\Lambda}_t \tilde{X}_n,\varphi\rangle = \langle\Lambda_t \tilde{\xi}_n \varphi\rangle$ have the same distribution.

Next let $\{\tilde{\xi}_{nk}\}$ be the subsequence corresponding to a rapidly increasing sequence $\{n_k\}$. Then

$$\sup_{0\le t\le 1}|\langle\Lambda\tilde{\xi}_{nk},\varphi\rangle(t) - \langle\Lambda\xi,\varphi\rangle(t)| \tag{4.2.15}$$

$$= \sup_{0\le t\le 1} \left|\varphi(t)X_{nk}(t) - \int_0^t \varphi'(s)X_{nk}(s)ds - (\varphi(t)W(t) - \int_0^t \varphi'(s)W(s)ds)\right|$$

$$\le \sup_{0\le t\le 1} |\varphi(t)(X_{nk}(t) - W(t))| + \sup_{0\le t\le 1} \left|\int_0^t \varphi'(s)(X_{nk}(s) - W(s))ds\right|$$

$$\longrightarrow 0 \quad \text{a.s.}$$

Since $\rho(x_{nk}, W) \to 0$ a.s. and $\max_{0\le t\le 1} |\varphi(t)| \le M$,

$\max_{0\le t\le 1} |\varphi'(t)| \le M$.

Next, let $\tilde{\xi}_{nk}$ be any subsequence of $\tilde{\xi}_n$. We can extract a further subsequence $\tilde{\xi}_{nk}$, such that $\{n_k'\}$ is a subsequence of $\{n_k\}$ and is increasing rapidly enough for (4.2.15) to hold. Hence, we have

$$\text{(4.2.16)} \qquad \sup_{0\le t\le 1} |\langle \Lambda\tilde{\xi}_n, \varphi\rangle(t) - \langle \Lambda\xi, \varphi\rangle(t)| \longrightarrow 0 \quad \text{a.s.}$$

ξ of course is the white noise corresponding to the Wiener process $W(t)$.

Convergence of the finite dimensional distributions is an immediate consequence of (4.2.16) since a.s. convergence is much stronger. This proves the theorem.

<u>Remark</u> : Observe that on taking φ to be a nonzero constant function, $\varphi' \equiv 0$ and we get Donsker's theorem on the convergence of random walks to the Brownian motion.

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We shall now obtain a simple approximation for the action of the discrete noise ξ_n and thus for white noise.

Observe that the process $X_n(t)$ has "jumps" of Y_i at the points $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$. Thus we can define the Stieltjes' integral

$$(4.2.17) \qquad \int_0^t \varphi(s)\,dX_n(s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \varphi(\tfrac{i}{n})\, Y_i \quad .$$

Summation by parts yields,

$$(4.2.18) \qquad \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \varphi(\tfrac{i}{n}) Y_i = \varphi(\tfrac{[nt]}{n}) \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} Y_i$$

$$- \sum_{i=1}^{[nt]} \{\varphi(\tfrac{i+1}{n}) - \varphi(\tfrac{i}{n})\}\{\frac{1}{\sqrt{n}} \sum_{j=1}^{i} Y_j\} \quad .$$

Using the Mean Value theorem, $\frac{i}{n} \le \alpha_i \le \frac{i+1}{n}$, the right hand side of the expression can be written,

$$(4.2.19) \qquad \varphi(\tfrac{[nt]}{n})\, X_n(t) - \sum_{i=1}^{[nt]} (\varphi'(\alpha_i)\tfrac{1}{n}) (\frac{1}{\sqrt{n}} \sum_{j=1}^{i} Y_i)$$

$$\doteq \varphi(t) X_n(t) - \int_0^t \varphi'(t) X_n(s)\,ds$$

approximately.

Thus $\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \varphi(\frac{i}{n}) Y_i$ is an approximation for the action $\langle \Lambda_t \xi_n, \varphi \rangle$ and hence also for the action $\langle \Lambda_t \xi, \varphi \rangle$ of white noise.

In the next section, we shall obtain a different approximation.

4.3 Noise Corresponding to Empirical Processes:

In this section we will define a noise phenomenon giving rise to empirical processes. From that we shall give an approximation for the action of "Brownian Bridge Noise" and hence also for white noise.

Recall that a "Brownian Bridge" $\{W^o(t): 0 \le t \le 1\}$ is a Gaussian process with mean 0 and covariance kernel

$$(4.3.1) \qquad R(s,t) = \text{Cov}(W^o(s), W^o(t)) = \min(s,t) - st .$$

Let $u_1, u_2, \ldots, u_n, \ldots$ be independent identically distributed random variables, distributed uniformly over $[0,1]$. That is to say

$$(4.3.2) \qquad P(u_i \le t) = t \ , \quad 0 \le t \le 1 \ .$$

We define the "Empirical process" $\{Y_n(t) : 0 \le t \le 1\}$ corresponding to the set $\{u_1, u_2, \ldots, u_n\}$ by

$$(4.3.3) \qquad Y_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{I(u_i \le t) - t\}$$

where $I(u_i \le t)$ is called the indicator random variable given by ,

$$\text{(4.3.4)} \qquad I(u_i \le t) = \begin{cases} 1\,, & u_i \le t \\ 0\,, & \text{otherwise} \end{cases}$$

It is well known that the process $Y_n(t)$ converges weakly to $W^o(t)$ in $D[0,1]$.

As before we define the operators $\Lambda_t Y_n$ and $\Lambda_t W^o$ by their actions on S, viz,

$$\text{(4.3.5)} \qquad \langle \Lambda_t Y_n, \varphi\rangle = \langle \Lambda Y_n, \varphi\rangle(t) = \int_0^t \varphi(s) Y_n(s)\,ds$$

$$\text{(4.3.6)} \qquad \langle \Lambda_t W^o, \varphi\rangle = \langle \Lambda W^o, \varphi\rangle(t) = \int_0^t \varphi(s) W^o(s)\,ds\ .$$

This enables us to define the corresponding noises, $\hat{\Lambda}_t Y_n = \Lambda_t \xi_n^o$ and $\hat{\Lambda}_t W^o = \Lambda_t \xi^o$ which we shall respectively call "Indicator Noise" and "Bridge Noise". Their actions are given by,

$$\text{(4.3.7)} \qquad \langle \hat{\Lambda}_t Y_n, \varphi\rangle = \langle \Lambda_t \xi_n^o, \varphi\rangle = \varphi(t) Y_n(t) - \int_0^t \varphi'(s) Y_n(s)\,ds$$

$$\text{(4.3.8)} \qquad \langle \hat{\Lambda}_t W^o, \varphi\rangle = \langle \Lambda_t \xi^o, \varphi\rangle = \varphi(t) W^o(t) - \int_0^t \varphi'(s) Y_n(s)\,ds.$$

A convergence theorem for $\hat{\Lambda}_t Y_n$, analogous to theorem (4.2.1) can be proved. However, we shall simply obtain an approximation.

Observe that $Y_n(t)$ has jumps of $\frac{1}{\sqrt{n}}$ at the random points $t = u_1, u_2, \ldots, u_n$. Thus we can write the Stieltjes' integral,

$$(4.3.9) \qquad \int_0^t \varphi(s)\,dY(s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \varphi(u_i)$$

where we have assumed without loss of generality, that $u_1 < u_2 < \ldots < u_n$.

Again, summation by parts yields as before the approximate identity:

$$(4.3.10) \qquad \langle \Lambda\xi_n^0 , \varphi\rangle(t) = \varphi(t)Y_n(t) - \int_0^t \varphi'(s)Y_n(s)ds$$

$$\doteq \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \varphi(u_i) \quad .$$

Thus, we can use (4.3.10) as an approximation for the action of indicator noise and hence for the action of bridge noise.

In this sense the action of $\Lambda\xi^0$ can be regarded as the limiting action of an average of "Random Dirac Measures".

Next consider the discrete process $X_n(t)$ considered in the last section. It is well known that the process $X_n(t) - tX_n(1)$ also converges weakly to the Brownian bridge. Formally we can say that the actions of "measures" $dX_n(t) - X_n(1)dt$ and $dY_n(t)$ are equivalent, by which we mean the actions of the corresponding noise operators are equivalent in distribution. Further, let us suppose that the process $X_n(t)$ is constructed from the same sequence $\{U_k\}$ as the process $Y_n(t)$. Then the action of the discrete noise ξ_n and that of white noise can be approximated by,

$$\text{(4.3.10)} \quad \int_0^t \varphi(s)\,dY_n(t) - X_n(1) \int_0^t \varphi(s)\,ds$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \varphi(U_i) - X_n(1) \int_0^t \varphi(s)\,ds$$

which is again the weighted average of random Dirac measures.

Remark : Integrals of the type (4.3.10) appear frequently in the asymptotic theory of rank statistics for example in the proof of the Chernoff-Savage theorem (also in our proof of theorem 2.2.1). Thus the asymptotic normality of linear rank statistics can be viewed as the result of the action of indicator noises of various types.

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